Queen Mary University of London

Group Theory

Week 12, Lecture 1, 2 & 3

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Sylow's Theorem 1

Theorem: Suppose G is a finite group and p is a prime. Then G has at least one Sylow p-subgroup.

Examples

Let $G = \mathcal{U}_9 = \{1, 2, 4, 5, 7, 8\}$, and $p = 3$. G is small enough that we can easily find a subgroup of order 3, but let's follow the proof of Sylow's Theorem 1.

Classification of p-Groups

Notation: Suppose G a finite group. Let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G, and let $\mathrm{n}_p\mathrm{G}$ denote the number of Sylow p-subgroups of G.

Classification of p-Groups

Take $G = \mathcal{D}_{10}$. Then a Sylow 5-subgroup is a subgroup of order 5.
One such subgroup is $\langle r \rangle$. In fact, this is the only example: the elements of $\mathcal{D}_{10} \setminus \langle r \rangle$ all have order 2, so cannot be contained in a subgroup of order 5. So a subgroup of order 5 is contained in $\langle r \rangle$, so must be $\langle r \rangle$. So $\mathrm{n}_2(\mathcal{D}_{10}) = 1$.
As a special case: if P is the only Sylow p-subgroup of G, then $P \trianglelefteq G$.

Classification of p-Groups

Proposition 7.9

Suppose G is a finite group and $P, Q \in \mathrm{Syl}_p G$ with $gQg^{-1} = Q$ for every $g \in P$. Then $P = Q$.

Classification of p-Groups

Sylow's Theorem 2, 7.10

Suppose G is a finite group and p is a prime. Then all the Sylow p-subgroups of G are conjugate.

Sylow's Theorems

Sylow's Theorem 3, 7.11

Suppose G is a finite group, and p is a prime, and write $|G| = p^a b$, where $p \nmid b$. Then $\mathrm{n}_p(G) \equiv 1 \mod p$, and $\mathrm{n}_p(G) \mid b$.

Sylow's Theorems

Remark

Sylow's Theorem 2 shows that if $P \in \mathrm{Syl}_p(G)$ and $P \trianglelefteq G$, then P is the only Sylow p-subgroup of G (because any other Sylow p-subgroup would have to be conjugate to P). In particular, if G is abelian, then (since all subgroups of an abelian group are normal) G has a unique Sylow p-subgroup.

Sylow's Theorems

Example

We can show that $\mathcal{C}_{15}$ is the only group of order 15 up to isomorphism.

Sylow's Theorems

Sylow's Theorems

Sylow's Theorems

Examples:

Suppose G is a group of order 20; then we claim that G cannot be simple.

Sylow's Theorems

Example: For a more complicated example, suppose G is a group of order 12; again we claim that G cannot be simple.

Sylow's Theorems

Sylow's Theorems

Sylow's Theorems

Exams Style Questions

Exams Style Questions

QMplus Quiz

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Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c, \; ac = ca = b, \; bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \dots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$