Queen Mary University of London

QE-WB-AJ (campus-M)

Group Theory

Week 11, Lecture 1, 2 & 3

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Final Exam U.G

6th January 10:00 – 12:30 pm

10:00 – 1:00 pm

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M. Sc

Final Exams papers of 2023 & 2024 will be discussed in Next week Lecture & Tutorial Time.

p-Groups

Definition: A p-group is a group whose order is a power of p.

for example, p^a.

Example: i) The cyclic group C_{p^r} is a p-group for any r.

ii) $V_4 = \{1, a, b, ab = c\}$ is a 2-group.

Examples

iii) $D_8 = D_{2^r}$ is a 2-group for any r.

D_{10}, D_{12} are not 2-group

Classification of p-Groups

Lemma

Suppose G is a p-group and $G \neq \{e\}$. Then $Z(G) \neq \{e\}$.

Solution: Suppose G is a p-group.

$|G| = p^n, \quad n > 0$

by the class equation

$$|G| = |Z(G)| + \frac{|G|}{Z(a_1)} + \cdots + \frac{|G|}{Z(a_k)}$$

Recall

$$p^n = |G| = [G : Z(a_i)]\,|Z(a_i)|$$

$$\frac{|G|}{|Z(a_i)|} = [G : Z(a_i)]$$ order of the conjugacy class of a_i.

$$p^n = |Z(G)| + p^{r_1} + \cdots + p^{r_k}$$

$p \mid |Z(G)|$ so $|Z(G)| > 1$

Classification of p-Groups

Lemma

Suppose G is a group of order p. Then $G \cong C_p$.

Proof: $g \neq$ in G, take g has order p.

So $\langle g \rangle$ is a subgroup of G of order p, so $G = \langle g \rangle \cong C_p$

$$G = \langle g \rangle = C_p = \{1, g, g^2, \ldots, g^{p-1}\}$$

Classification of p-Groups

Proposition

Suppose G is a group of order p^2. Then G is isomorphic to $\mathcal{C}_{p^2}$ or $\mathcal{C}_p \times \mathcal{C}_p$.

Solution: The order of any element of G is 1, p, or p^2.

If there is an element h of order p^2

Then $G = \langle g \rangle \cong C_{p^2}$

Suppose there is no element of order p^2

$\Rightarrow$ every element other than 1 is of order p.

Now, we can find an element $h \neq 1 \in Z(G)$. $H = \langle h \rangle$, then H is normal subgroup of G of order p.

$k \in G \setminus H$ & $K = \langle k \rangle$, then

$H < HK \leq G$, but then by Lagrange's Theorem

$G = HK$

$$G = \{ h^i k^j \mid 0 \leq i, j < p \}$$

with $\operatorname{ord}(h) = \operatorname{ord}(k) = p$

& $hk = kh$

One can define an isomorphism

$$G \longrightarrow C_p \times C_p$$

$$h^i k^j \longmapsto (z^i, z^j)$$

Classification of p-Groups

$h \mapsto k$

$k \mapsto 8$

$k^{-1} = h^{-1} = h^3$

Proposition 7.4

There are exactly five groups of order p^3 up to isomorphism. If $p = 2$, they are

$hk = kh$ $\quad k^2 = 1,\ h^4 = 1$

$$\underline{\mathcal{C}_8}, \quad \underline{\mathcal{C}_4 \times \mathcal{C}_2}, \quad \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2, \quad \underline{\mathcal{D}_8}, \quad \mathcal{Q}_8.$$

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_2 \right\} \leq GL_3(\mathbb{F}_2)$$

$p^3 = 2^3 = 8$

G is a group of order 8.

So must be isomorphic to one of

$C_8,\ C_4 \times C_2,\ C_2 \times C_2 \times C_2,\ D_8,\ Q_8$

Classification of p-Groups

G is non-abelian

H cannot be isomorphic to C_8

H must have an element of order 4.

$$h = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cc} a & b \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \neq$$

$$\begin{array}{cc} b & a \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}$$

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$$H = \langle h \rangle = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$=$$

$$\langle h \rangle = \{1, h, h^2, h^3\}$$

$$k \in G \setminus H \qquad k = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

$$k^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$hk \neq kh$

This should be true fo C_8.

$r^{-1}s = sr$

$h^{-1}s = kh$

$h^3 s = kh$

$$G \cong D_8$$

$$h \mapsto r, \quad k \mapsto s$$

$$\boxed{G \cong C_4 \times C_2 \qquad kh = hk.}$$

Classification of p-Groups

Example

Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_2 \right\}$$

Then G s a group of order 8, so must be isomorphic to one of the groups in previous Proposition 7.4. Let's follow the proof above to find out which one.

We have seen above that G is isomorphic to D_8

and cannot be isomorphic to C_8
as its non-abelian.

Not isomorphic to $C_4 \times C_2$
as $k^2 \neq h^2$ $kh \neq hk$

Not isomorphic to Q_8 Quaternion
as $k^2 \neq h^2$

Simple p- groups and composition series

Proposition 7.5

Suppose G is a p-group and $G \neq \{e\}$. Then G has a normal subgroup of order p. Hence the only simple p-group is C_p.

Solution: $h \neq 1 \in Z(G)$ $= h^{p^{a-1+1}} = h^{p^a} = 1$

ord(h) is a power of p; Say p^a $\left(h^{p^{a-1}}\right)^p =$

$\Rightarrow h^{p^{a-1}}$ has order p

$\Rightarrow \langle h^{p^{a-1}} \rangle$ is a subgroup of order p.

Normal subgroup. $h \in Z(G)$

Simple p- groups and composition series

Corollary 7.6

Suppose G is a group and $|G| = p^n$. Then the composition factors of G are $\mathcal{C}_p, \ldots, \mathcal{C}_p$ (n- copies).

$$G_0 \rhd \cdots \rhd G_r$$

$$\frac{G_0}{G_1}, \ldots, \frac{G_{r-1}}{G_r} \cong C_p$$

p-subgroups

$|G| = p^a b \qquad p \nmid b$

Definition

Suppose G is a finite group and p is a prime, and write the order of G as $p^a b$, where p does not divide b. A Sylow p-subgroup of G is a subgroup of order p^a.

Examples: $G = C_{100}$ $\qquad |G| = 2^2 \times 5^2$

$\langle g^{25} \rangle$ is a Sylow 2-subgroup

$\langle g^{25} \rangle = \{1, g^{25}, g^{50}, g^{75}\} \qquad |g^{25}| = 4 = 2^2$

$\langle g^{4} \rangle = \{1, g^{4}, g^{8}, g^{12}, \ldots\}$ $\qquad \langle g^4 \rangle$ is a Sylow 5-subgroup.

p-subgroups

Examples: $G = S_3$ $\qquad |G| = 2 \times 3$

$$S_3 = \{ e,\ (12),\ (13),\ (23),\ (123),\ (132) \}$$

$H = \{(12), e\}$ is a Sylow 2-Subgroup.

$K = \{(13), e\}$ " "

$H' = \{(23), e\}$ " " "

p-subgroups

Examples:

$L = \{(123), e\}$ is a Sylow 3-subgroup.

$\underline{\cancel{(132)}}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$G = D_{12}$

$|G| = \underline{2^2} \times \underline{3}$

p-subgroups

$$\langle r \rangle = \{1, r, r^2, r^3, r^4, r^5\}$$

$$\langle r^2 \rangle \leq \langle r \rangle$$

$$\langle r^2 \rangle = \{1, r^2, r^4\}$$

$\langle r^2 \rangle$ is a Sylow 3-Subgroup of D_{12}.

$\langle r^2, s \rangle = \{1, s, r^2 s, r^4 s\}$ is a Sylow 2-Subgroup of D_{12}

p-subgroups

Sylow's Theorem I:

Suppose G is a finite group, and p is a prime. Then G has at least one Sylow p-subgroup.

Example: $G = U_9 = \{1, 2, 4, 5, 7, 8\}$

$p = 3$

p-subgroups

Proof: $|G| = p^a b$, where $p \nmid b$.

Let X be the set of all subsets of G of size p^a

$$X = \left\{ \underset{|p^a}{|X_1|}, \underset{p^a}{|X_2|}, \dots \dots X_n \right\}$$

we have an action λ of G on X by

$$\left(\lambda_g \left\{ (\underline{s_1, \dots s_{p^a}}) \right\} = \left\{ g s_1, g s_2, \dots g s_{p^a} \right\} \right)$$

$|X| = \binom{p^a b}{p^a}$ is not divisible p $\left(\dfrac{p^a b!}{p^a! \, (p^a b - p^a)!} \right)$

$\Rightarrow$ there must be an orbit Y of $\mathcal{S}$ that $|Y|$ is not divisible by p.

$\mathcal{S} = \{S_1, \dots, S_{p^a}\} \in Y \quad P = \underline{Stab(S)}$

$P \leq G$, $\quad |Y|\,|\underline{P}| = |G| = p^a b$

Since $|Y|$ is not divisible by $p \Rightarrow |P|$ is divisible by p^a.

$|P| \leq p^a$

$|P| = p^a$

P is a Sylow p-Subgroup.

Example: $G = U_9 = \{1, 2, 4, 5, 7, 8\}$ $\quad p = 3$

Let X be the set of 3-Subsets of U_9

$\{a, b, c\}$ $\quad abc$ $\quad |X| = \binom{6}{3} = \frac{6!}{3!\,3!} = \frac{\cancel{6}\times5\times4\times\cancel{3!}}{\cancel{3!}\times\cancel{3}\times\cancel{2}} = 20$

20 which is not divisible by 3.

$X = \{124, 248, 478, 125, 178, 125, \ldots\}$

$Orb(124) = \{124, 248, 487, 517, 578, 125\}$

$Orb(248) = \{\ \ldots\ \}$

$Orb(127) = \{147, 258\}$ not divisible by 3

$Y = \{147, 258\}$

$H = \{147\}$

$Stab(H) = P$

$P = \{1, 4, 7\}$ Slow 3-Subgroup

Composition Series

Jorda-Holder Theorem

Suppose G is a group, and that G has two composition series

$$G_0 \lhd G_1 \lhd \cdots \lhd G_r \quad \text{and} \quad H_0 \lhd H_1 \lhd \cdots \lhd H_s \lhd \{1\}.$$

Then $r = s$ and the groups

$$\frac{G_0}{G_1}, \ldots, \frac{G_{r-1}}{G_r}$$

are isomorphic to the groups

$$\frac{H_0}{H_1}, \ldots, \frac{H_{r-1}}{H_r}$$

in some order.

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Definition: G is a finite group.

Let $Syl_p(G)$ denote the set of Sylow p-subgroups of G & let $n_p(G)$ denote the no of Sylow's p-subgroups of G.

Composition Series

Tutorial - Session.

we are covering past
Exams papers 2023- 2024
during Tutorial Times.

Exams Style Questions

QMplus Quiz

There will be no Quiz this week, we ll be discussing Exam Style Questions in Tutorials.

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2 s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$