Queen Mary University of London

Mth6106: Group Theory (Mid-term)

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This coursework counts for 20% of your mark for this module. You should answer all questions, and each question will be marked out of 10. You should give full explanation of your answers. Please submit your solutions on QMPlus by 5pm on Friday 15th November 2024. Your submission must be entirely your own work.

Question 1 [15 marks].

(a) Suppose that G is a group. Let $f, g \in G$ and suppose that $\mathrm{ord}(f) = 3$ and $\mathrm{ord}(g) = 6$.

- (i) What is $\mathrm{ord}(f^2)$? **[3]**
- (ii) What is $\mathrm{ord}(g^3)$? **[2]**

(b)
- (i) Give example of a finite group of order at least 8 which is not abelian. (You do not need to prove that your examples have the properties claimed.) **[2]**
- (ii) Find two groups of countably infinite order which are not isomorphic to one another. **[2]**

(c) Suppose G is a group and $g \in G$. Prove that the inverse of g is unique. **[2]**

(d) A boolean group B is a group such that $g^2 = e$ for every $g \in B$, where e is the identity elemenet of the group B. Prove that every boolean group is abelian. **[2]**

(e) With reference to the group axioms, give a reason why it is **not** possible to complete the following table in a way which results in the Cayley table of a group:

	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a			1		
b	b	1	c	e	a	d
c	c				1	
d	d		1	a	b	
e	e		a			1

[2]

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Question 2 [25 marks].

(a) This question is about the dihedral group $\mathcal{D}_{10}$, the group of symmetries of a regular pentagon.

(i) Describe all the elements of $\mathcal{D}_{10}$ (draw a diagram like we did for $\mathcal{D}_8$ in lectures, and label the axes of reflections and mention the rotations in terms of r and s). **[2]**

(ii) Let $H = \langle r^2 s\rangle \leq \mathcal{D}_{10}$. Give a list of all **left** cosets of H in $\mathcal{D}_{10}$. (Hint: cosets generated by r, etc). **[3]**

(b) Write down the possible cycle types that an element of $\mathcal{S}_5$ can have. Which of these cycle types occur for elements of $\mathcal{A}_5$? **[5]**

(c) Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 5 & 6 & 7 & 8 & 4 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 6 & 5 & 1 & 7 & 8 & 2 \end{pmatrix}.$$

are elements of $\mathcal{S}_8$.

(i) Write down orders of f and g. **[2]**

(ii) Check if f and g are conjugate in $\mathcal{S}_8$? Give reason. **[1]**

(iii) If they are conjugate find out the value of k such that $kfk^{-1} = g$. **[2]**

(d) Let G be a group and H is a normal subgroup of G. Prove that

$$Hg = gH \quad \forall g \in G$$

[5]

(e) Find all the elements in the conjugacy class of r^3 in $\mathcal{D}_{10}$. Show your complete working. **[5]**

