Queen Mary University of London

Mth6106: Group Theory (Mid-term Solutions)

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Question 1 [15 marks].

(a) **Solution**: Suppose that G is a group. Let $f, g \in G$ and suppose that $\mathrm{ord}(f) = 3$ and $\mathrm{ord}(g) = 6$.

(i) Since $\mathrm{ord}(f) = 3$ we have $f^2 \neq 1$. (If we had $f^2 = 1$ then $\mathrm{ord}(f)$ would have to be 1 or 2.) We also have $(f^2)^2 = f^4 = f \cdot f^3 = f \neq 1$ since f has order 3 and does not have order 1. On the other hand $(f^2)^3 = f^6 = f^3 \cdot f^3 = 1 \cdot 1 = 1$, so the smallest integer $k \geq 1$ such that $(f^2)^k = 1$ is $k = 3$. Thus $\mathrm{ord}(f^2) = 3$. [**3**]

(ii) Since $\mathrm{ord}(g) = 6$ we have $g^3 \neq 1$, so $\mathrm{ord}(g^3) \neq 1$. On the other hand $(g^3)^2 = g^6 = 1$ so $\mathrm{ord}(g^3)$ must be 2. [**2**]

(b) (i) **Solution**: (Seen) Probably $\mathcal{D}_8$ is the most obvious example, but students could also choose $\mathcal{A}_n$ or $\mathcal{S}_n$ with $n \geq 4$. [**2**]

(ii) **Solution**: Two countably infinite groups which are not isomorphic to each other. $\mathbb{Q}$ and $\mathbb{Z}$ are good examples.(1 mark for each infinite group) [**3**]

(c) **Solution**: We first show that every element of B is its own inverse. Certainly, if $g = e$ then the result is clear. Otherwise,

$g.g = e$ so that, by uniqueness, $g^{-1} = g$.

If $g, h \in B$ then it follows that $(gh) = (gh)^{-1} = h^{-1}g^{-1} = hg$. [**2**]

(d) **Solution.** (Seen.) We have $db = 1$ but $bd = a$, which contradicts the axiom of inverses (called axiom G4 in the notes). It is also acceptable to note that since $ba = 1$ we must have $ab = 1$ by the axiom of inverses, which would lead to 1 appearing twice in the same row. (5 marks for solution and 5 marks for justification) [**3**]

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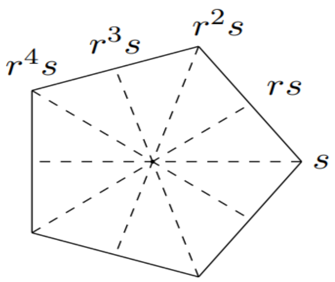
Question 2 [25 marks].

(a) (i) **Solution**: The dihedral group of order 10, denoted as $\mathcal{D}_{10}$, is the group of symmetries of a pentagon, consisting of rotations r (r is a rotation by $72°$ so $r^5 = 1$) of order 5 and reflections s of order 2 considering as five axes of symmetry. It has 10 elements: 5 rotations and 5 reflections.

[**4**]



$$\mathcal{D}_{10} = \{1, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\}$$

(ii) **Solutions**: The set $\langle r^2 s\rangle$ is generated by the element r^s is a combination of a rotation (r^2) and a reflection (s) in D_{10}. Since $r^5 = e$ and $s^2 = e$ we can generate the elements of H by taking successive powers of r^2s.

$$\langle r^2 s\rangle = \{e, r^2 s\} \leq \mathcal{D}_{10}$$

the subgroup itself a coset: $eH = \{e, r^2s\}$
left coset from r: $rH = \{r, r^3s\}$
left coset from r^2: $r^2H = \{r^2, r^4s\}$
left coset from r^3: $r^3H = \{r^3, s\}$ left coset from r^4: $r^4H = \{r^4, rs\}$ [**3**]

(b) How many types of conjugacy classes does $\mathcal{S}_5$ have?

Solutions: Recall that the **cycle type** of f is just the list of the lengths of the cycles of f when written in cycle notation, in decreasing order. Since every element of $\{1,2,3,4,5\}$ appears in exactly one cycle, the cycle type is therefore just a list of positive integers in decreasing order that add up to 5. So we have

$$(1,1,1,1,1), (2,1,1,1), (2,2,1), (3,1,1), (3,2), (4,1), (5).$$

For f to be in $\mathcal{A}_5$, we have to have ev$f$ even. So the possible cycle types for $\mathcal{A}_5$ are the cycle types for $\mathcal{S}_5$ with an even number of even entries, i.e. $(1,1,1,1,1)$, $(3,1,1)$, $(2,2,1)$ and (5). [**5**]

(c) Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 & 5 & 6 & 7 & 8 & 4 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 6 & 5 & 1 & 7 & 8 & 2 \end{pmatrix}.$$

are elements of $\mathcal{S}_8$. Write down orders of f and g.

(i) Write down orders of f and g. [**3**]

Solution: We have $f = (123)(45678)$ and $g = (145)(23678)$. Order of f and g is 15.

(ii) Check if f and g are conjugate in $\mathcal{S}_8$? Give reason. [**2**]

Solution: Yes, since the cycle type of of f and g is same. f is conjugate to g.

(iii) If they are conjugate find out the value of k such that $kfk^{-1} = g$. [**3**]

Solution: We can find k by mapping $1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 2$ and so on as follows:

$$k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 5 & 2 & 3 & 6 & 7 & 8 \end{pmatrix}.$$

such that $kfk^{-1} = g$.

(d) (i) Apply the Subgroup Test to HN.

$1 \in H$ and $1 \in N$, so $1 = 1.1 \in HN$.

Given two elements of HN, we can write them as hm and kn for $h, k \in H$ and $m, n \in N$. Then $hk^{-1} \in H$ and $mn^{-1} \in N$, so by the normal condition $k(mn^{-1})k^{-1} \in N$ as well. So

$$(hm)(kn)^{-1} = hmn^{-1}k^{-1} = hk^{-1}kmn^{-1}k^{-1} \in HN.$$

[**5**]