Group Theory

Week 11, Lecture 1, 2 & 3

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p-Groups

Definition: A p-group is a group whose order is a power of p.

Examples

Classification of p-Groups

Lemma

Suppose G is a p-group and $G \neq \{e\}$. Then $Z(G) \neq \{e\}$.

Classification of p-Groups

Lemma

Suppose G is a group of order p. Then $G \cong \mathcal{C}_p$.

Classification of p-Groups

Proposition

Suppose G is a group of order p^2. Then G is isomorphic to $\mathcal{C}_{p^2}$ or $\mathcal{C}_p \times \mathcal{C}_p$.

Classification of p-Groups

Proposition 7.4

There are exactly five groups of order p^3 up to isomorphism. If $p = 2$, they are

$$\mathcal{C}_8, \quad \mathcal{C}_4 \times \mathcal{C}_2, \quad \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2, \quad \mathcal{D}_8, \quad \mathcal{Q}_8.$$

Classification of p-Groups

Classification of p-Groups

Example

Let

$$G =$$

Then G s a group of order 8, so must be isomorphic to one of the groups in previous Proposition 7.4. Let's follow the proof above to find out which one.

Simple p- groups and composition series

Proposition 7.5

Suppose G is a p-group and $G \neq \{e\}$. Then G has a normal subgroup of order p. Hence the only simple p-group is $\mathcal{C}_p$.

Simple p- groups and composition series

Corollary 7.6

Suppose G is a group and $|G| = p^n$. Then the composition factors of G are $\mathcal{C}_p, \ldots, \mathcal{C}_p$ (n- copies).

p-subgroups

Definition

Suppose G is a finite group and p is a prime, and write the order of G as $p^a b$, where p does not divide b. A Sylow p-subgroup of G is a subgroup of order p^a.

p-subgroups

Examples:

p-subgroups

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p-subgroups

p-subgroups

p-subgroups

Composition Series

Jorda-Holder Theorem

Suppose G is a group, and that G has two composition series

$$G_0 \lhd G_1 \lhd \cdots \lhd G_r \qquad \text{and} \qquad H_0 \lhd H_1 \lhd \cdots \lhd H_s \lhd \{1\}.$$

Then $r = s$ and the groups

$$\frac{G_0}{G_1}, \ldots, \frac{G_{r-1}}{G_r}$$

are isomorphic to the groups

$$\frac{H_0}{H_1}, \ldots, \frac{H_{r-1}}{H_r}$$

in some order.

Composition Series

Composition Series

Example:

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Exams Style Questions

Exams Style Questions

QMplus Quiz

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$