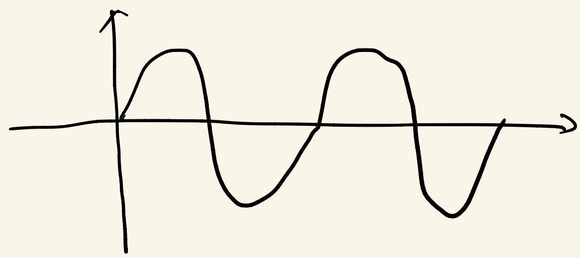
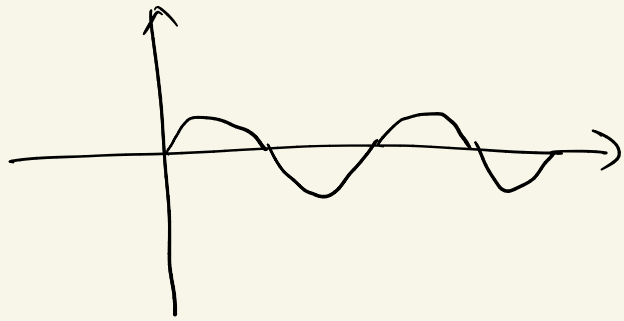
Selected solutions to PS 10

1. The rough behaviour of u over time is as follows

$t=0$

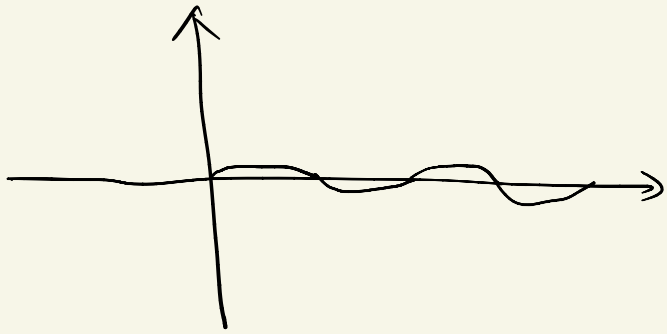


$t=1$



$\vdots$

t large



2. $\frac{d}{dt}\int_0^L u(x,t)\,dx$

$= \int_0^L \frac{d}{dt}[u(x,t)]\,dx$

$= \int_0^L u_t(x,t)\,dx$

$= \int_0^L k u_{xx}(x,t)\,dx$ ← by the PDE

$$= k u_x \Big|_0^L$$

$$= k u_x(L,t) - k u_x(0,t)$$ ← by the Neumann boundary condition

$$= 0 - 0$$

$$= 0$$

so $\int_0^L u(x,t)\,dx$ is a conserved quantity.

3. If u does not depend on t, the PDE becomes

$$0 = u_t = k u_{xx}$$

so

$$u_{xx} = 0$$

$$u_x = c$$

$$u = cx + d \quad \text{a linear function}$$

If the temperature distribution does not change with time, it has to be a linear function.

4. Let us consider $u(x,t) = e^{-bt} V(x,t)$

namely $V(x,t) = e^{bt} u(x,t)$

we have

$$V_x(x,t) = e^{bt} u_x(x,t)$$

$$V_{xx}(x,t) = e^{bt} u_{xx}(x,t)$$

and $V_t(x,t) = be^{bt}u(x,t) + e^{bt}u_t(x,t)$

So we get

$$V_t - kV_{xx} = be^{bt}\cdot u + e^{bt}\cdot u_t - ke^{bt}\cdot u_{xx}$$

$$= e^{bt}\cdot[u_t - ku_{xx} + bu]$$

by the PDE satisfied by u. $\rightarrow$

$$= e^{bt}\cdot 0$$

$$= 0$$

So V satisfied the equation

$$\begin{cases} V_t - kV_{xx} = 0 \\ V(x,0) = e^{0\cdot t}u(x,0) = f(x) \end{cases}$$

By Fourier-Poisson formula, we get

$$V(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\cdot f(y)\,dy$$

and so

$$u(x,t) = e^{-bt}V(x,t)$$

$$= \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\cdot f(y)\,dy$$

5. By the maximum principle the max/min happen on $t=0$, $x=0$ or $x=1$

at $t=0$ $\quad u = x^2$

so min at $(x,t) = (0,0)$, min value 0

max at $(x,t) = (1,0)$, max value 1.

at $x=0$, $\quad u = 2kt$

so min at $(x,t) = (0,0)$, min value 0

max at $(x,t) = (0,T)$, max value $2kT$

at $x=1$, $\quad u = 1+2kt$

so min at $(x,t) = (0,0)$, min value 1

max at $(x,t) = (1,T)$, max value $1+2kT$.

Combining these, we have globally

maximum at $(1,T)$, maximum value $1+2kT$

minimum at $(0,0)$, minimum value 0.