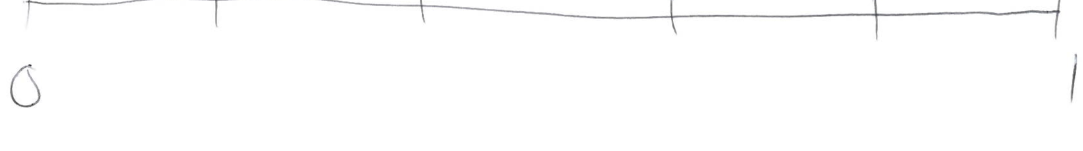
Dimension

There are various ways of thinking about the 'dimension' of a geometric object A $(\subseteq \mathbb{R}^d)$.

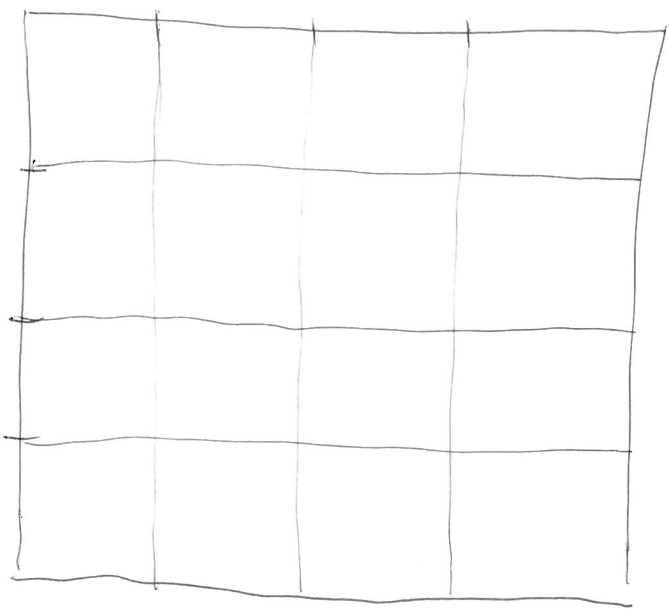
A fruitful approach stems from the following simple observation:

- For $A = [0,1] \subseteq \mathbb{R}$, if we divide A up into sub-intervals of length $\varepsilon = 1/k$ then there are $k = k^1$ such intervals.

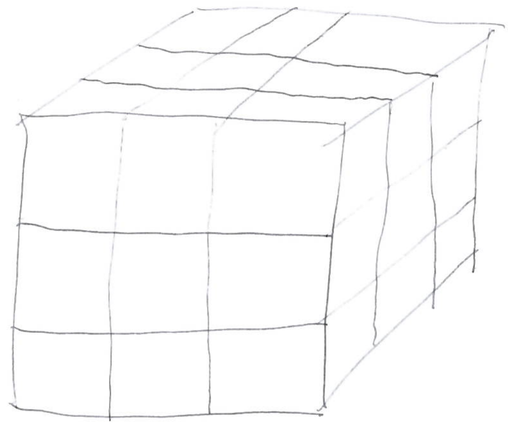


- For $A = [0,1]^2 = \{(x,y) : x, y \in [0,1]\}$
$$\subseteq \mathbb{R}^2$$

If we divide A up into sub-squares of side length $\varepsilon = 1/k$ then there are k^2 such sub-squares



- For $A = [0,1]^3 \subset \mathbb{R}^3$ (the unit cube), if we divide up into sub-cubes of side length $\varepsilon = 1/k$ then there are k^3 such sub-cubes



For $A = [0,1]^d = \underbrace{[0,1] \times [0,1] \times \ldots \times [0,1]}_{d \text{ times}}$,

let $N(\varepsilon) = N(1/k) = k^d$ be the number of 'sub-cubes' of side length $\varepsilon = 1/k$ needed to fill A.

Intuitively, $A = [0,1]^d$ is a d-dimensional object, and the value d can be extracted / recovered by considering the quantity

$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log(k^d)}{\log k}$$

$$= \frac{d \log(k)}{\log k} = d$$

<u>Idea</u>: Extend this thinking to define dimension of more general sets.

Box dimension ('Box counting dimension')

For $d \in \mathbb{N}$, a <u>d-dimensional cube</u> (or 'box') of side length ε is a subset of $\mathbb{R}^d$ of the form

$$[a_1, a_1+\varepsilon] \times [a_2, a_2+\varepsilon] \times \dots \times [a_d, a_d+\varepsilon]$$

$$\left(= \{ (x_1, \dots, x_d) : a_i \leq x_i \leq a_i + \varepsilon \text{ for all } 1 \leq i \leq d \} \right)$$

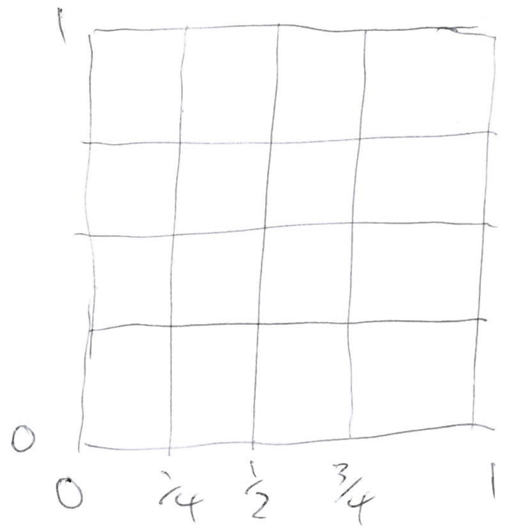
[Such cubes have volume ε^d]

Let A be a subset of $\mathbb{R}^d$.

Let $N(\varepsilon)$ denote the smallest number of cubes of side length ε needed to <u>cover</u> A.

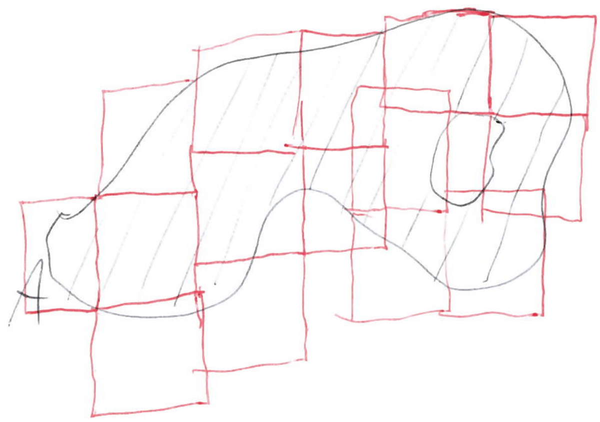
This means that the union of these cubes is larger than A, i.e. the union contains A as a subset.

e.g.



Here $N(1/4) = 16$

e.g.



Definition The <u>box dimension</u> (or <u>box counting dimension</u>) of A is given by:

$$D(A) := \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

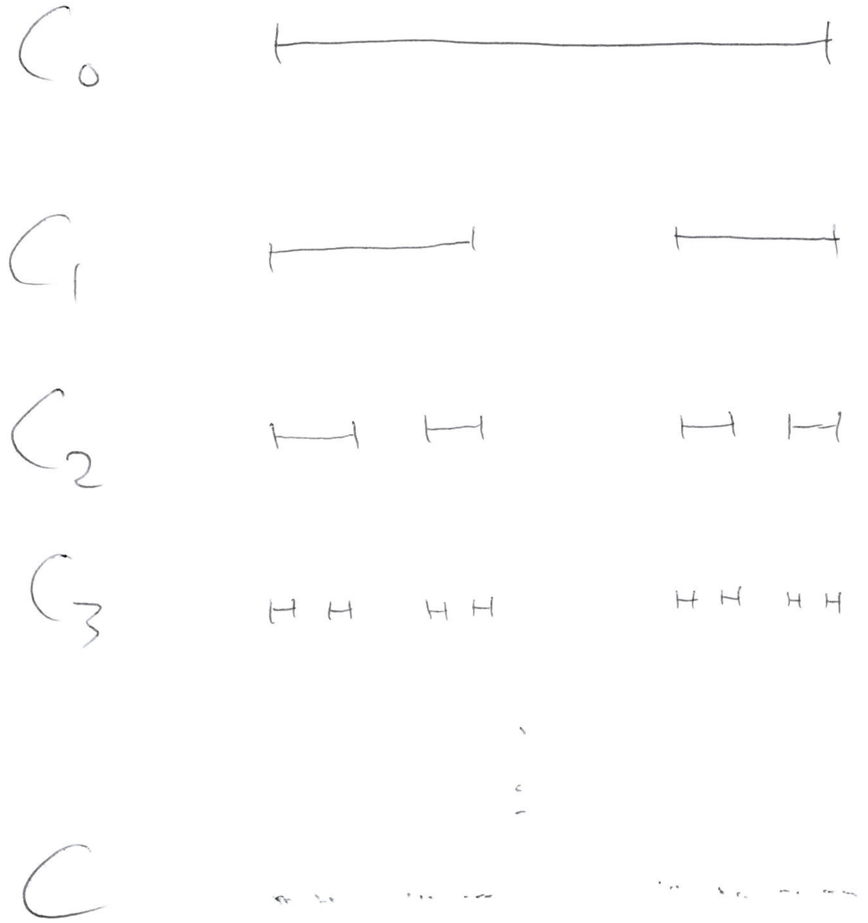
$$= \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon}$$

(if this limit exists)

<u>Idea</u> : $$N(\varepsilon) \sim \left(1/\varepsilon\right)^{D(A)}$$

<u>Defn</u> The set A is called a <u>fractal</u> if $D(A)$ is not an integer.

<u>Middle - Third Cantor Set</u> C



$$C = \bigcap_{n=0}^{\infty} C_n$$

If $\varepsilon = 1/k = 1/3^n$ then C is a subset of C_n, for any $n \geqslant 0$

i.e. The collection of length-$1/3^n$ intervals making up C_n covers C, in the sense that their union contains C.

Since C_n consists of 2^n intervals, each of length $1/3^n = \varepsilon$,

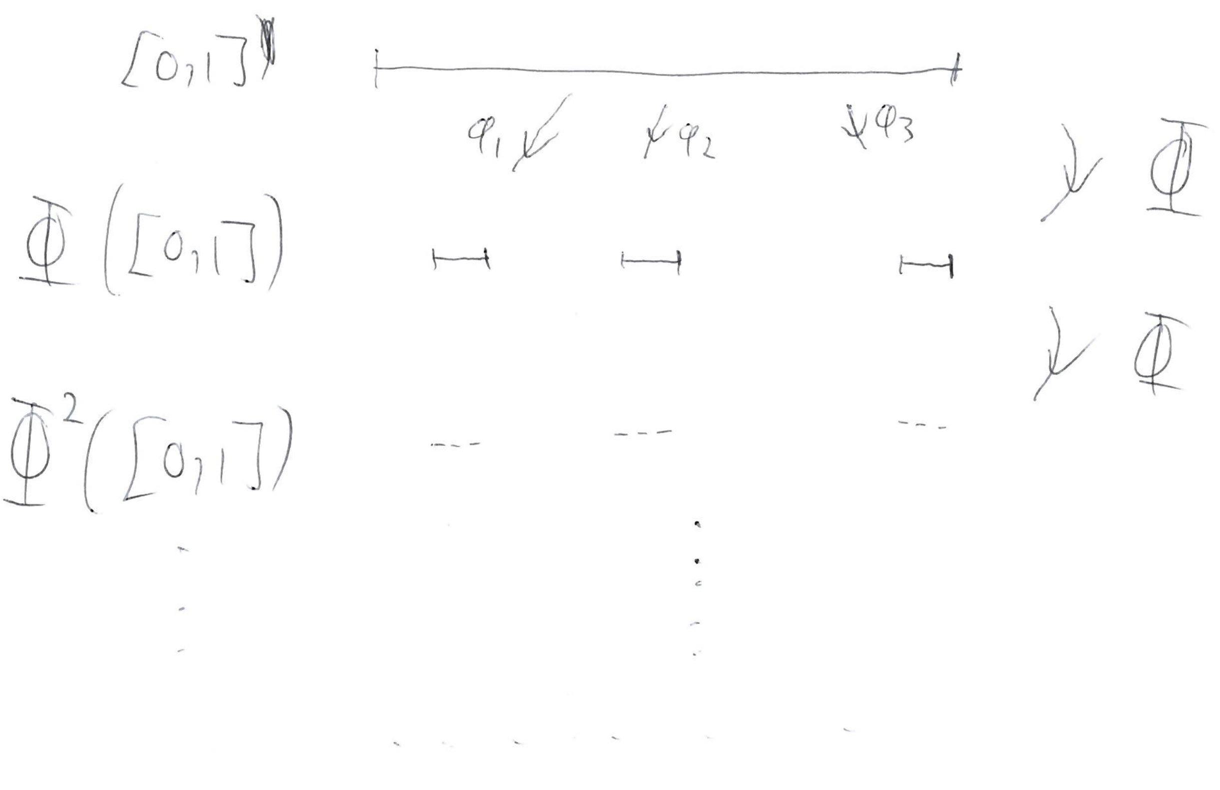
then $N(\varepsilon) = 2^n$

So $$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log(2^n)}{\log(3^n)} = \frac{n \log 2}{n \log 3}$$

$$= \frac{\log 2}{\log 3} \simeq 0.631\ldots$$

The middle-$\frac{1}{3}$ Cantor set C has box dimension $D(C) = \frac{\log 2}{\log 3}$, which is not an integer, hence C is a fractal.

<u>Example</u> (cf. Example of the set K of numbers in $[0,1]$ with decimal expansions only using digits $\{1,4,9\}$)



Recall that the Cantor set K is given by $K = \bigcap_{n=0}^{\infty} \Phi^n([0,1])$

Let $\varepsilon = 1/k = \frac{1}{10^n}$

Note $\Phi^0([0,1]) \supset \Phi^1([0,1]) \supset \Phi^2([0,1])$

$\supset \dots$

so $K \subset \Phi^n([0,1])$ for all $n \geq 0$.

In other words, K is covered by the 3^n disjoint closed intervals that make up $\Phi^n([0,1])$. Each of these intervals is of length $\frac{1}{10^n} = \varepsilon$.

Then $N(\varepsilon) = N(1/10^n) = 3^n$

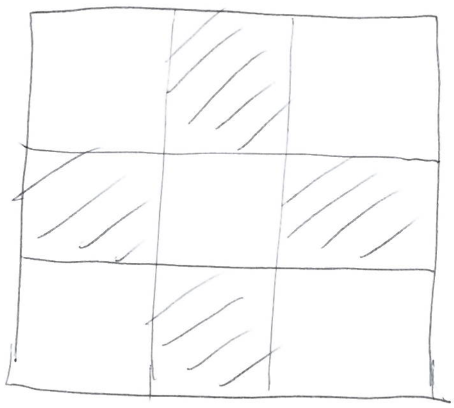
Then $$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log 3^n}{\log(10^n)}$$

$$= \frac{n \log 3}{n \log 10}$$

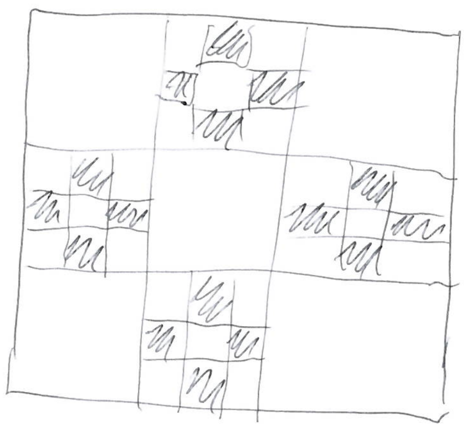
$$= \frac{\log 3}{\log 10} \simeq 0.4771\ldots$$

This is the box dimension of K, and since $D(K) = \frac{\log 3}{\log 10}$ is not an integer, then K is a fractal.

Example ('Checkerboard' in $\mathbb{R}^2$)



F_1



F_2

Let $F = \bigcap_{n=0}^{\infty} F_n$.

Let $\varepsilon = 1/k = 1/3^n$

Each $F_n = \Phi^n([0,1]^2)$ is the union of 4^n squares, each of side length $1/3^n$.

Then $$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log(4^n)}{\log(3^n)}$$

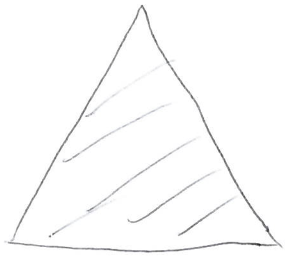
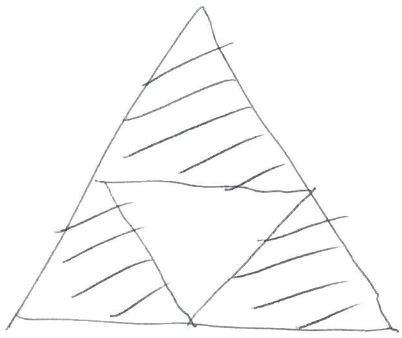
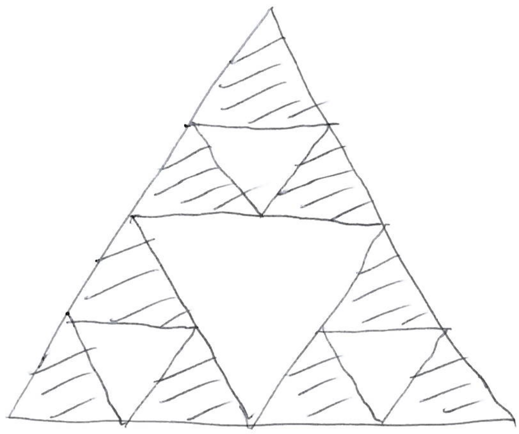
$$= \frac{n \log 4}{n \log 3}$$

$$= \frac{\log 4}{\log 3} \simeq 1.26\ldots$$

So the box dimension of F is

$$D(F) = \frac{\log 4}{\log 3},$$ and since this is not an integer then F is a fractal.

The Sierpinski Triangle

n	P_n
0	
1	
2	

$$\vdots$$

The Sierpinski Triangle is defined to be

$$P = \bigcap_{n=0}^{\infty} P_n$$

Let us express the Sierpinski triangle in the language of iterated function systems.

For $j = 1, 2, 3$,

let $z_j^* \in \mathbb{R}^2 \equiv \mathbb{C}$

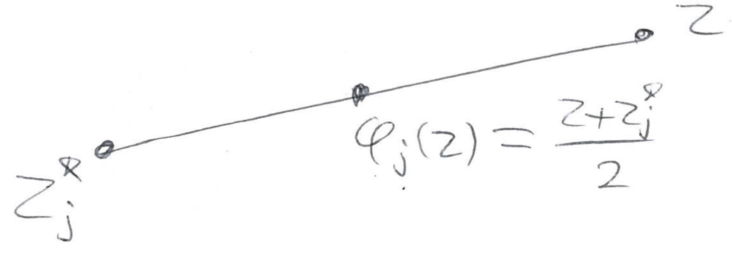
This identification is purely for convenience of notation

be three points which are <u>not</u> collinear,

and define $\varphi_j : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \cong & & \cong \\ \mathbb{C} & & \mathbb{C} \end{array}$

by $\varphi_j(z) = \dfrac{z + z_j^*}{2} = \dfrac{z}{2} + \dfrac{z_j^*}{2}$

(so $\varphi_j(z)$ is the mid-point between z and z_j^*)



Clearly $\varphi_j(z_j^*) = z_j^*$

i.e. z_j^* is a fixed point of φ_j

(in fact it is the unique fixed point)

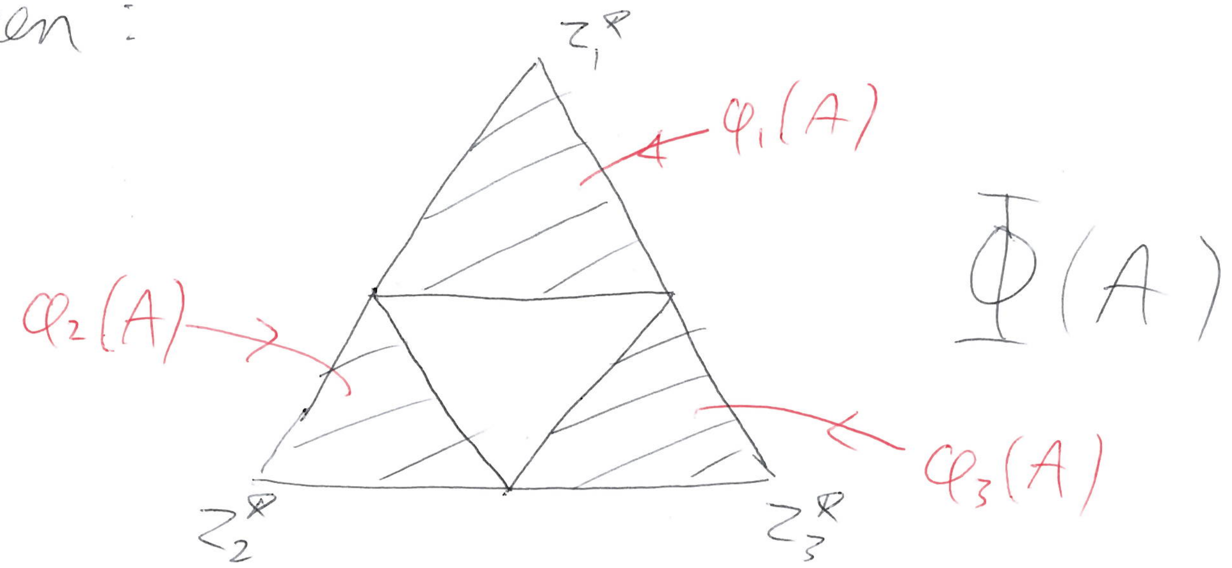
[Moreover, z_j^* is an attracting fixed point for φ_j, and its basin of attraction is the whole of $\mathbb{R}^2 \equiv \mathbb{C}$]

The maps $\varphi_1, \varphi_2, \varphi_3$ constitute an iterated function system, and we can (as usual) associate the map Φ defined by

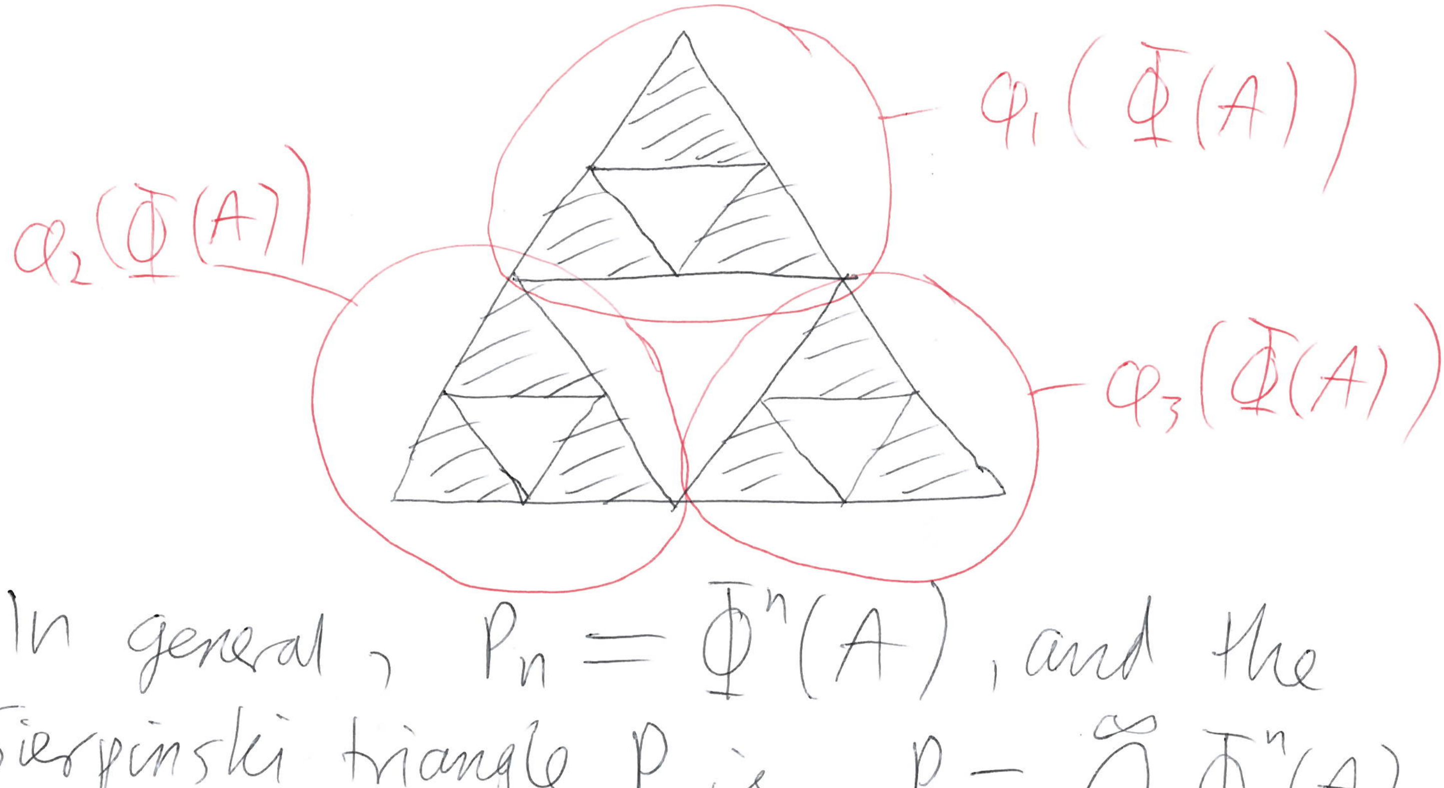
$$\Phi(A) = \bigcup_{j=1}^{3} \varphi_j(A)$$

for all $A \subseteq \mathbb{R}^2 \equiv \mathbb{C}$

If for example A is the solid triangle with vertices z_1^*, z_2^*, z_3^*, then:



$$\downarrow \Phi$$



In general, $P_n = \Phi^n(A)$, and the Sierpinski triangle P is $P = \bigcap_{n=0}^{\infty} \Phi^n(A)$

<u>Note</u> In the various preceding examples (middle-third Cantor set, $\{1,4,9\}$ decimal digit example, checkerboard example) the recursive rule was the <u>same</u> at each step, and there two <u>scaling factors</u> appearing in each step:

- Side length of 'boxes'

$$\varepsilon_{n+1} = \alpha \, \varepsilon_n$$

(where $0 < \alpha < 1$)

- Number of 'boxes' $N_{n+1} = \beta N_n$

(where $\beta > 1$ is an integer)

Thus $\varepsilon_n = \varepsilon_0 \alpha^n$

$$N_n = N_0 \beta^n$$

This allows a nice formula for the box dimension D of the various examples:

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

$$= \lim_{n \to \infty} \frac{\log N_n}{\log(1/\varepsilon_n)}$$

$$= \lim_{n \to \infty} \frac{\log N_n}{-\log(\varepsilon_n)}$$

$$= \lim_{n \to \infty} \frac{\log(N_0 \beta^n)}{-\log(\varepsilon_0 \alpha^n)}$$

$$= \lim_{n \to \infty} \frac{\log N_0 + \log(\beta^n)}{-\log(\varepsilon_0) - \log(\alpha^n)}$$

$$= \lim_{n \to \infty} \frac{\log N_0 + n \log \beta}{-\log(\varepsilon_0) - n \log \alpha}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\operatorname{Log} N_0 + \operatorname{Log}\beta}{-\frac{1}{n}\operatorname{Log}(\varepsilon_0) - \operatorname{Log}\alpha}$$

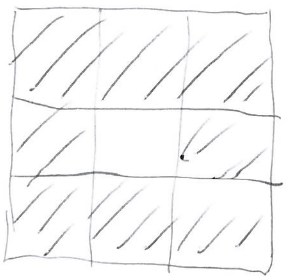
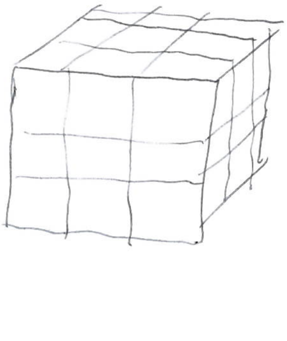
$$= \frac{\lim_{n\to\infty}\left(\frac{1}{n}\operatorname{Log} N_0 + \operatorname{Log}\beta\right)}{\lim_{n\to\infty}\left(-\frac{1}{n}\operatorname{Log}(\varepsilon_0) - \operatorname{Log}\alpha\right)}$$

$$= \frac{0 + \operatorname{Log}\beta}{-0 - \operatorname{Log}\alpha}$$

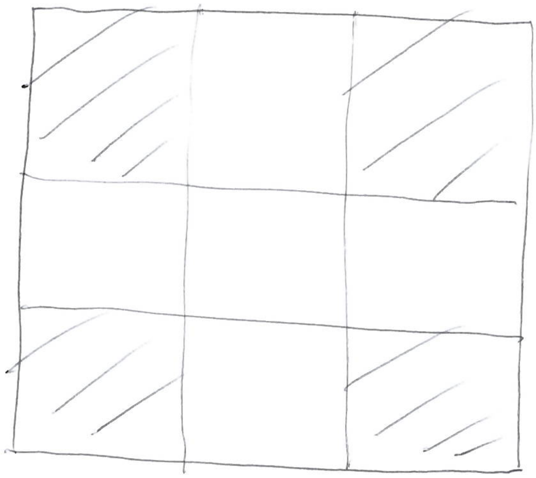
$$= \frac{\operatorname{Log}\beta}{-\operatorname{Log}\alpha}$$

$$= \frac{\operatorname{Log}\beta}{\operatorname{Log}(1/\alpha)}$$

We can now give various (other) box dimensions:

Set	α	β	Dimension D
Middle-$\frac{1}{3}$ Cantor set	$\frac{1}{3}$	2	$\frac{\log 2}{\log 3}$
Sierpinski Triangle	$\frac{1}{2}$	3	$\frac{\log 3}{\log 2}$
Sierpinski Carpet 	$\frac{1}{3}$	8	$\frac{\log 8}{\log 3}$
Menger Sponge 	$\frac{1}{3}$	20	$\frac{\log 20}{\log 3} \simeq 2 \cdot 7 \ldots$

Example (Similar to checkerboard)



$$\varphi_1(x,y) = \left(\tfrac{x}{3}, \tfrac{y}{3}\right)$$

$$\varphi_2(x,y) = \left(\frac{x+2}{3}, \tfrac{y}{3}\right)$$

$$\varphi_3(x,y) = \left(\tfrac{x}{3}, \frac{y+2}{3}\right)$$

$$\varphi_4(x,y) = \left(\frac{x+2}{3}, \frac{y+2}{3}\right)$$

Let $\Phi(A) = \bigcup_{i=1}^{4} \varphi_i(A)$

Let $F_k = \Phi^k([0,1]^2)$

Then $F := \bigcap_{k=0}^{\infty} F_k$ has box dimension

$$D = D(F) = \frac{\log 4}{\log(1/\tfrac{1}{3})} = \frac{\log 4}{\log 3}$$

$\left(= 2 \times \frac{\log 2}{\log 3} = 2 \times D(C),\right.$

where C is the middle-$\tfrac{1}{3}$ Cantor set$\left.\right)$

Note that $F = C \times C = C^2$

(where C is the middle-third Cantor set)

So we see that

$$\begin{aligned} D(F) &= D(C \times C) \\ &= D(C) + D(C) \\ &= 2\,D(C) \end{aligned}$$

<u>Example</u> If we <u>also</u> introduce

$$\varphi_5(x,y) = \left(\frac{x}{3}, \frac{y+1}{3}\right)$$

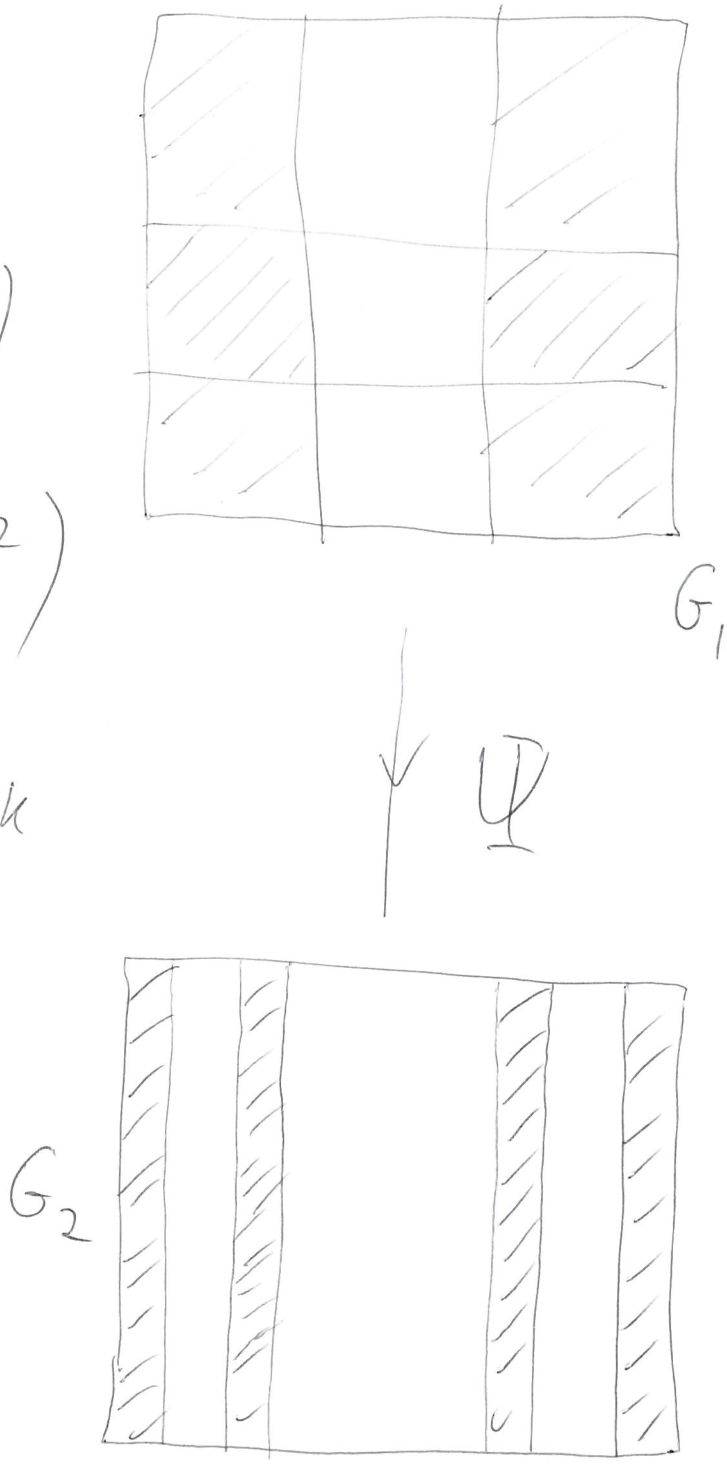
$$\varphi_6(x,y) = \left(\frac{x+2}{3}, \frac{y+1}{3}\right)$$

Now define Ψ

by $\Psi(A) = \bigcup_{i=1}^{6} \varphi_i(A)$

and $G_k = \Psi^k\left([0,1]^2\right)$

and $G = \bigcap_{k=0}^{\infty} G_k$



Note: $G = C \times [0,1]$,

where C is the middle-$\frac{1}{3}$ Cantor set

The box dimension of G is

$$D(G) = \frac{\log 6}{\log(1/\tfrac{1}{3})} = \frac{\log 6}{\log 3}$$

$$= \frac{\log(2.3)}{\log 3}$$

$$= \frac{\log 2 + \log 3}{\log 3}$$

$$= \frac{\log 2}{\log 3} + 1$$

$$= D(C) + D([0,1])$$

(ie. $D(C \times [0,1]) = D(C) + D([0,1])$)