PROBLEM SET 10 FOR MTH 6151

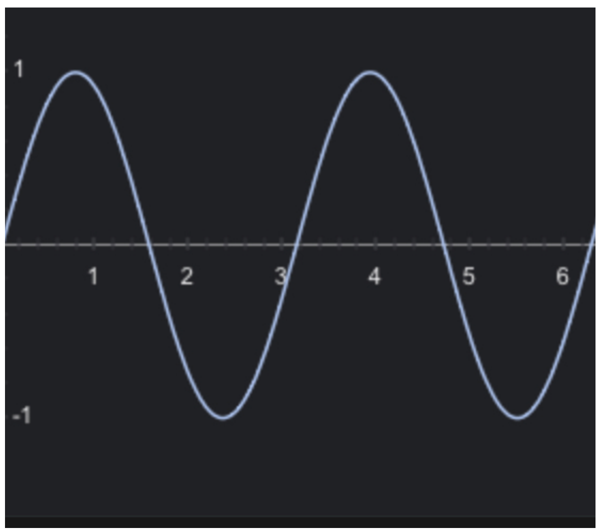
1. Describe in qualitative terms the behaviour of the solution to the heat equation on an interval

$$U_t = \varkappa U_{xx}, \qquad x \in [0, 2\pi],$$

with initial data

$$U(x, 0) = f(x)$$

where $f(x)$ has the form



and

$$U(0, t) = U(2\pi, t) = 0.$$

What do you expect to be the limit of $U(x, t)$ as $t \to \infty$? No proof or calculations are required. You may draw a plot of the solution at various instants of time to explain your answer.

2. Suppose U solves the following heat equation on the interval with Neumann boundary conditions

$$U_t = \varkappa U_{xx}, \qquad x \in [0, L], \qquad t \geq 0,$$
$$U(x, 0) = f(x),$$
$$U_x(0, t) = U_x(L, t) = 0.$$

Show that

$$\int_0^L U(x, t) dx$$

is a conserved quantity, i.e. its time derivative being zero.

3. Find the general solution to the heat equation

$$U_t = \varkappa U_{xx}$$

in the case that $U = U(x)$ —that is, when U does not depend on the coordinate t. What the interpretation of this result?

4. CHALLENGE: Solve the heat equation with constant dissipation

$$\begin{aligned} &U_t - \varkappa U_{xx} + bU = 0, \qquad x \in \mathbb{R}, \\ &U(x,0) = f(x), \end{aligned}$$

where b is a constant. HINT: consider the change of variables $U(x,t) = e^{-bt}V(x,t)$.

5. Use even extensions to find the solution to the problem on the half-line with Neumann boundary conditions

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x > 0, \quad t > 0, \\ &U(x,0) = f(x), \\ &U_x(0,t) = 0. \end{aligned}$$

6. Consider the solution

$$U(x,t) = x^2 + 2\varkappa t$$

of the heat equation. Find the location of its maximum and minimum in the rectangle

$$\{0 \le x \le 1,\ 0 \le t \le T\}.$$