WEEK 11 NOTES

1. The heat equation on the half-line

As an application of the Fourier-Poisson formula we now study the initial value problem for the heat equation on the half-line with Dirichlet boundary conditions. More precisely, we have

$$U_t = \varkappa U_{xx}, \qquad x \geq 0, \quad t > 0, \tag{1.1}$$

$$U(x,0) = f(x), \tag{1.2}$$

$$U(0,t) = 0 \qquad \text{(Dirichlet boundary condition)}. \tag{1.3}$$

We have studied a similar problem for the wave equation. These require the use of *odd extensions*. Recall that the odd extension of f is defined as

$$F(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}.$$

The point behind the use of odd extensions is that it allows one to formulate an *auxiliary initial value problem* on the whole real line —namely,

$$V_t = \varkappa V_{xx}, \qquad x \in \mathbb{R},$$
$$V(x,0) = F(x).$$

The solution to this problem is given in terms of the Fourier-Poisson formula

$$\begin{aligned} V(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy. \end{aligned}$$

Observe, however, that the integrand in the second equation is odd, so setting $y \mapsto -y$ (so that $dy \mapsto -dy$) in the first integral gives

$$\begin{aligned} V(x,t) &= -\frac{1}{\sqrt{4\pi\varkappa t}} \int_{\infty}^{0} e^{-\frac{(x+y)^2}{4\varkappa t}} F(-y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{\infty}^{0} e^{-\frac{(x+y)^2}{4\varkappa t}} F(y)dy + \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} F(y)dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} \left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}} \right) F(y)dy. \end{aligned}$$

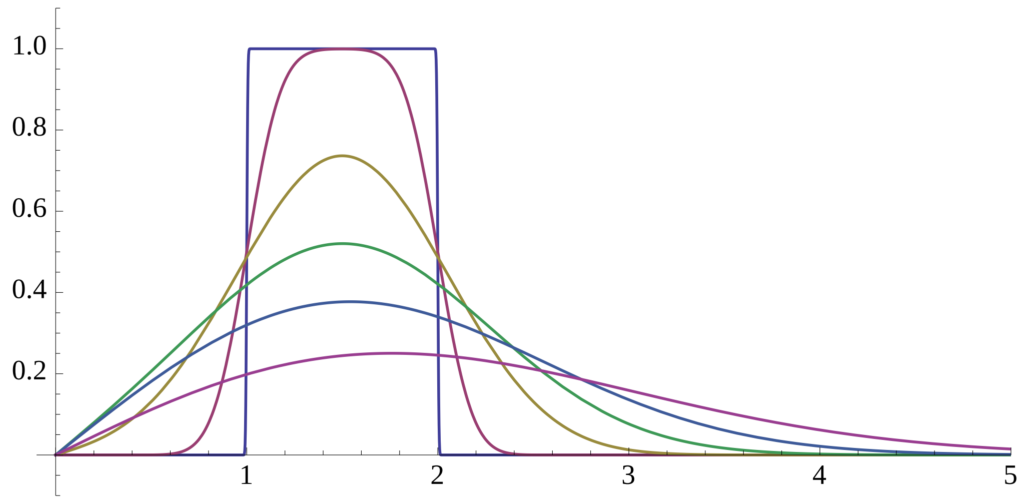
Now, recalling that $F(y) = f(y)$ for $x \in [0,\infty)$ we find that:

$$V(x,t) = \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} \left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}} \right) f(y)dy. \tag{1.4}$$

Now, observe that by construction one readily has that

$$V(0,t) = \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} \left(e^{-\frac{y^2}{4\varkappa t}} - e^{-\frac{y^2}{4\varkappa t}} \right) f(y)dy = 0.$$

Thus, the function V as given by equation (1.4) satisfies the boundary conditions for $U(x,t)$. Thus, if the solution $U(x,t)$ to (1.1)-(1.3) is *unique* (something we have not proved!) then $U(x,t)$ and $V(x,t)$ must coincide for $x>0$, $t>0$. A plot of the solution (1.4) for various times for an initial function f with the shape of a top hat function is given below:



It is worth pointing out that

$$\tilde{K}(x,t) \equiv \frac{1}{\sqrt{4\pi\varkappa t}}\left(e^{-\frac{(x-y)^2}{4\varkappa t}} - e^{-\frac{(x+y)^2}{4\varkappa t}}\right)$$

for y fixed is also a solution to (1.1)-(1.3) with initial condition a Dirac delta centred at $x=y$. We call $\tilde{K}(x,t)$ the fundamental solution to the heat equation with Dirichlet boundary conditions.

Note. *The case of Neumann boundary conditions (where $U_x(0,t)=0$) can be studied in a similar manner using even extensions —see Coursework 10.*

2. The maximum/minimum principle for the heat equation

We conclude our discussion of the heat equation with a discussion of some general properties of the heat equation.

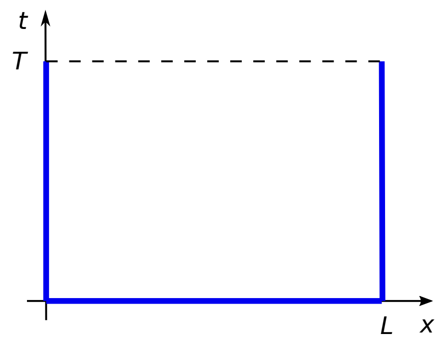
2.1. **Maximum principle on space interval.** We first consider the maximum principle in a finite rectangular space-time region.

Theorem 2.1. *If $U(x,t)$ satisfies the heat equation on the* space-time rectangle

$$\Omega = \left\{0 \le x \le L, \;\; 0 \le t \le T\right\}$$

then the maximum value of $U(x,t)$ is either assumed initially (i.e. at $t=0$) or on the boundaries $x=0$ or $x=L$.

The region in which one has to look for the maxima/minima of the solution to the heat equation are highlighted in blue in the figure below:



Remark 2.2. The minimum of $U(x,t)$ on Ω satisfies a similar property.

Example 2.3. Find the maxima and minima of

$$U(x,t) = 1 + e^{-\varkappa t}\cos x$$

on the rectangle

$$\Omega = \Big\{ -\pi \leq x \leq \pi, \;\; 0 \leq t \leq T \Big\}$$

for some $T > 0$. It can be directly verified that this is a solution to the heat equation.

Following the maximum/minimum principle for the heat equation one then has to look for the maximum/minimum values of U on the sides $t = 0$, $x = -\pi$ and $x = \pi$:

(i) **On** $t = 0$**.** One has that

$$U(x,0) = 1 + \cos x.$$

As $\cos x$ ranges between -1 and 1 if $x \in [-\pi,\pi]$ it follows that the maximum of $U(x,0)$ $x \in [-\pi,\pi]$ occurs at $x = 0$ where $U(x,0)$ takes the value 2 and the minima at $x = -\pi$, π where $U(x,0)$ takes the value 0.

(ii) **On** $x = -\pi$**.** Here one has

$$U(0,t) = 1 - e^{-\varkappa t}.$$

It can be verified that the derivative of this function is positive for $t \geq 0$. Thus, the maximum must happen at $t = T$ (where U takes the value $1 - e^{-\varkappa T} < 1$) and its minimum at $t = 0$ (where U takes the value 0).

(iii) **On** $x = \pi$**.** Here one has

$$U(0,t) = 1 - e^{-\varkappa t}.$$

Thus, again, the maximum must happen at $t = T$ (where U takes the value $1 - e^{-\varkappa T} < 1$) and its minimum at $t = 0$ (where U takes the value 0).

So, putting together the information above one has that the maximum of U occurs at $(0,0)$ where it takes the value 2 and the minima, which takes the value 0, occur at $(-\pi,0)$ and $(\pi,0)$.

In the following we give a sketch of the proof of the above Theorem —it illustrates the applicability of some ideas of Calculus.

Proof. The rectangle Ω is finite region (a *bounded set*) so that the function $U(x,t)$ should attain a maximum and a minimum somewhere.

If the maximum occurs in the interior of Ω at a point (x_*, t_*) one then has that

$$U_t(x_*, t_*) = U_x(x_*, t_*) = 0.$$

This is the standard condition of the vanishing of the gradient at an extremal point. In addition, as one has a maximum then the *second derivative test* has to hold —i.e. one has that

$$U_{xx}(x_*, t_*) < 0. \tag{2.1}$$

However, the function $U(x,t)$ satisfies the heat equation $U_t = \varkappa U_{xx}$. Thus, in addition one has that

$$U_{xx}(x_*, t_*) = 0.$$

The latter is in contradiction with (2.1). Thus, the maximum cannot occur in the interior of Ω. It can only be attained somewhere on the sides (boundary) of the rectangle.

Next, assume that the maximum is attained somewhere, (x_*, T) $x \in (0, L)$, in the middle of the top of the rectangle. The restriction of U to the top of the rectangle, $U(x, T)$ $x \in (0, L)$, is a function of x only. At the maximum one has that

$$U_x(x_*, T) = 0, \qquad U_{xx}(x_*, T) < 0.$$

Now, as (x_*, T) is a maximum over the whole of Ω one has that

$$U_t^- \equiv \lim_{t \to T} U_t(x_*, t) \geq 0.$$

Again, the heath equation gives

$$0 \geq U_t(x_*, T) = \varkappa U_{xx}(x_*, T) < 0.$$

This, again, is a contradiction. Hence, the maximum cannot be attained at the top of the rectangle —it can only be attained on the bottom or on the sides.

□

Remark 2.4. The argument above is a sketch of the actual proof. A full proof needs to exclude some *pathological situations* which we have overlooked for the sake of conciseness.

Remark 2.5. The argument cannot be used to exclude the bottom of the rectangle as the heat equation is only solved for $t > 0$.

2.2. Maximum principle for heat type equations on the real line, stability and uniqueness.

We again restrict ourselves to solutions with finite total heat. i.e.

$$\int_{-\infty}^{\infty} |U(x,t)| dx < \infty.$$

Theorem 2.6. *If U satisfies*

$$U_t - \varkappa U_{xx} = 0$$

and $U(x, 0) \leq M$, then we must have

$$U(x,t) \leq M, \quad \forall t.$$

i.e. the maximum is attained at initial time.

Remark 2.7. By flipping the sign, we have the minimum is also attained at the initial time.

As a fist application, we show the following "No Butterfly Effect" for heat type equations. It says if the initial error is small, then the error for later time cannot be too large

Proposition 2.8. *[Stability] If U satisfies the following heat type equations*

$$U_t - \kappa U_{xx} + bU = 0,$$

and $U(x, 0) \leq M$, then we have

$$U(x, t) \leq Me^{-bt}.$$

in particular $U(x, T) \leq C_{M,T}$ is bounded by a constant depending on the time T and initial maximum M.

Proof. We consider $V(x, t)$ defined by $V(x, t) = U(x, t)e^{bt}$. Then $U = Ve^{-bt}$ plugged into the equations becomes

$$V_t e^{-bt} - bVe^{-bt} - \varkappa V_{xx} e^{-bt} + bVe^{-bt} = 0.$$

Namely

$$V_t - \varkappa V_{xx} = 0$$

satisfies the heat equation. The maximum principle on the real line Theorem 2.6 gives

$$V \leq M.$$

So $U = Ve^{-bt} \leq Me^{-bt}$. □

As a second application, we also have the following uniqueness property

Proposition 2.9. *If U_1 and U_2 are solutions to the following initial value problem*

$$\begin{cases} U_t - \varkappa U_{xx} + bU = 0 \\ U(x, 0) = f \end{cases}$$

Then we have

$$U_1 \equiv U_2.$$

Proof. We see by principle of superpositions that $W = U_1 - U_2$ satisfies $W_t - \varkappa W_{xx} + bW = 0$, the stability Proposition 2.8 applied to $M = 0$ tells us $U_1 - U_2 = W \leq 0$.

Similarly we can show $U_2 - U_1 \leq 0$ by applying Proposition 2.8 to $\tilde{W} = U_2 - U_1$. Thus

$$U_1 \equiv U_2.$$

□