Main Examination period 2024 – January – Semester A

MTH6106: Group Theory

Examiners: I.D. Morris, F. Rincón

You will have a period of **3 hours** to complete the exam. You have additional **30 minutes** for scanning and submitting your solution.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [25 marks].

(a) For the following, either give an example or explain why no example can exist:

  (i) A group with at least four elements in which every element has order either 1 or 2. **[2]**

  (ii) A group with at least four elements in which every element has order either 1 or 4. **[2]**

  (iii) Two groups of order 24 which are not isomorphic to one another. **[2]**

  (iv) Two countably infinite groups which are not isomorphic to each other. **[2]**

(b) Let $G = \{x \in \mathbb{R} \colon x \geq 0\}$ and define a binary operation $\circ$ on G by $x \circ y := |x - y|$. Decide which of the four group axioms are satisfied by $(G, \circ)$ and which are not. For each axiom, give a brief justification for your answer. **[5]**

(c) Using Lagrange's theorem, or otherwise, show that if g is an element of a group G such that $|G| = n$, then g^n is the identity element of G. **[3]**

(d) Using the result of (c) above, show that if p is a prime number and n is an integer in the range $1 \leq n \leq p$, then $n^{p-1} \equiv 1 \mod p$. (Hint: consider the group $\mathcal{U}_p$.) **[3]**

(e) List all subgroups of the dihedral group $\mathcal{D}_{10}$ and indicate briefly why your list is complete. **[6]**

Question 2 [25 marks].

(a) Consider the two permutations $f, g \in S_5$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}.$$

  (i) Write each of f, g, and fg in disjoint cycle notation. **[5]**

  (ii) State the **order** of each of f, g, and fg. Briefly justify your answer with reference to a result from the course. **[3]**

  (iii) State the **cycle type** of each of f, g and fg. **[3]**

  (iv) Which of f, g and fg are **conjugate** to one another? Briefly justify your answer with reference to a result from the course. **[3]**

  (v) Which of f, g and fg are elements of the **alternating group** A_5? Briefly justify your answer with reference to a result from the course. **[3]**

(b) (i) Consider the element r^3 of the dihedral group $\mathcal{D}_{10}$. Find the **centraliser** of r^3 in $\mathcal{D}_{10}$. Give a brief justification for your answer. **[3]**

  (ii) Now instead consider the element r^3 of the dihedral group $\mathcal{D}_{12}$. Find the **centraliser** of r^3 in $\mathcal{D}_{12}$. Give a brief justification for your answer. **[3]**

(iii) Write down the **centre** of the group $\mathcal{D}_{10}$ and give a brief justification for your answer. **[2]**

Question 3 [25 marks].

(a) Give an example of a group G and subgroup $H \leq G$ such that H is **not** normal in G. **[2]**

(b) Show that:

(i) If N is a normal subgroup of an abelian group G, then G/N is also abelian. **[3]**

(ii) If $\phi\colon G \to H$ is a group homomorphism and G is abelian, then $\operatorname{im}\phi$ is abelian. **[3]**

(c) Using the Third Isomorphism Theorem, or otherwise, prove that if H_1 and H_2 are subgroups of an abelian group G, then

$$|H_1 H_2| = \frac{|H_1| \cdot |H_2|}{|H_1 \cap H_2|}.$$

Indicate clearly in your answer where, and how, you make use of the fact that G is abelian. **[4]**

(d) Define

$$G := \mathcal{U}_{56,}, \qquad H_1 := \{1, 5, 9, 13, 25, 45\}, \qquad H_2 := \{1, 3, 9, 17, 19, 25\}.$$

Calculate $|H_1 H_2|$. **[4]**

(e) Let $\mathbb{F}_4$ denote the field with four elements, let $\mathbb{F}_4^2$ denote the set of all two-dimensional vectors with entries in $\mathbb{F}_4$, and recall that $\mathrm{GL}_2(\mathbb{F}_4)$ denotes the group of invertible 2×2 matrices with entries in $\mathbb{F}_4$. In this question we consider the action π of $\mathrm{GL}_2(\mathbb{F}_4)$ on $\mathbb{F}_4^2$ defined by $\pi(A, v) := Av$.

(i) How many elements of $\mathrm{GL}_2(\mathbb{F}_4)$ belong to the stabiliser of the vector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$? **[3]**

(ii) How many vectors belong to the orbit of the vector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$? **[3]**

(iii) Using an appropriate result from the course, calculate the order of $\mathrm{GL}_2(\mathbb{F}_4)$. **[3]**

Question 4 [25 marks].

(a) Suppose that G is a group of order 117. Prove that G cannot be simple. State carefully any additional results from the course which you require in your answer. **[5]**

(b) Suppose that G is a group of order 168, and let n_7 denote the number of Sylow 7-subgroups of G. List all possible values that n_7 could take which are consistent with Sylow's theorems. Is it possible to decide, using this list of values, whether or not G is simple? Why, or why not? **[4]**

(c) Without proof, write down:

(i) A composition series for the group $\mathcal{C}_{27}$. [**2**]

(ii) A composition series for the group $\mathcal{D}_{12}$. [**2**]

(iii) A composition series for the group $\mathcal{S}_4$. [**3**]

(iv) A composition series for one of the three groups listed in (i)–(iii) above which is **different** to the composition series which you stated in your earlier answer. [**2**]

(d) Show that if a group is abelian then all of its inner automorphisms are trivial. [**2**]

(e) Let p be a prime number and let G be the group of integers $\{0, 1, \ldots, p-1\}$ equipped with the binary operation of addition modulo p. By considering the effect of each automorphism on the element 1, show that the outer automorphism group of G has exactly $p-1$ elements. [**5**]

End of Paper.