Queen Mary
University of London

Group Theory

Week 10, Lecture 1, 2 & 3

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Campus-M- PO-PT-TK

6-th January 10:00 am- 12:30pm

Final Exams Dates: 80% weitage

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$G \times X \longrightarrow X$ ✓

Group Actions

Suppose G is a group and X is a set. An action of G on X is a collection $\pi = (\pi_g \,|\, g \in G)$ of functions from X to X such that:

Student's Explain.

1. $\pi_1 = \mathrm{id}_X$, and $\quad \pi_1(x) = x$
2. $\pi_f \circ \pi_g = \pi_{fg}$ for all $f, g \in G$.

$$\pi_f \circ \pi_g(x) = \pi_f(g(x)) = f(g(x)) = (fg)(x) = \pi_{fg}$$

G, a group, X, a set

$\pi : G \longrightarrow X$

$A_1)$ $\pi_1 = id$

$A_2)$ $\pi_f \circ \pi_g = \pi_g \circ \pi_f$

$\forall f, g \in X.$

Group Actions

$GL_2(\mathbb{R})$

1) **Exercise.** Let $G = GL(2, \mathbb{R})$ and $X = \mathbb{R}^2$.

(1) Show that the map

$$G \times X \to X, \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

defines a G action.

(2) What are the orbits and fixed point sets of this G action?

$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

$M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

The collection of all invertible matrices constitutes the general linear group $GL(2, \mathbb{R})$.

1) $G \times X \to X$ as given defines an action

A1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

A2) $\pi_{M_1} \pi_{M_2} \begin{pmatrix} x \\ y \end{pmatrix} = M_1 \left(M_2 \begin{pmatrix} x \\ y \end{pmatrix} \right) = M_1 \begin{pmatrix} a_2 x + b_2 \\ c_2 x + d_2 \end{pmatrix}$

Group Actions

$$= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 x + b_2 \\ c_2 x + d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 x + a_1 b_2 f \\ b_1 c_2 x + b_1 d_2 \\ c_1 a_2 x + c_1 b_2 f \\ d_1 c_2 x + d_1 d_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

2) **Exercise.** Let $H \leq G$, and define H action by restricting the map $H \times X$. Calculate the orbits and fixed point sets in the following cases:

(1) $H = SO(2)$.

(2) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(3) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(4) $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$.

(5) $H = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$.

2)

$$O_x = \{ gx : g \in G \}$$

$$\underset{2\times 2}{M} \cdot \underset{2\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

$M \in GL_2(\mathbb{R}$ $\qquad M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$M = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$ if $x \neq 0$

& $M = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$ if $y \neq 0$

$$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \times 1 + 0 \times 0 \\ y \times 1 + 1 \times 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \times 1 + 0 \\ y \times 1 + 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbb{R}^2 \setminus \{0\} = G \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

fixed point: $M \cdot 0 = \{0\}$

The origin is the only fixed point.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times x + 0 \\ 0 \times x + 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Group Actions

$$H = SO(2) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : 0 \le \varphi \le 2\pi \right\}$$

$$G = GL_2(\mathbb{R}) \qquad H \le GL_2(\mathbb{R})$$

$$\left.\begin{array}{l} G \times X \to X \\ G \times \mathbb{R}^2 \to \mathbb{R}^2 \end{array}\right\} \qquad H \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos\varphi + 0 \times \sin\varphi \\ r\sin\varphi + 0 \times \cos\varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$M_1 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

Check that this defines an action from $H \times \mathbb{R}^2 \to \mathbb{R}^2$

A_1 : $I\begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$

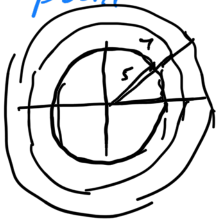
A_2 : $M_1 M_2 \begin{pmatrix} r \\ 0 \end{pmatrix} = M_1\left(M_2 \begin{pmatrix} r \\ 0 \end{pmatrix}\right) = \left(M_1 M_1 \begin{pmatrix} r \\ 0 \end{pmatrix}\right)$

$= \mathbb{R}^2$

2) $H \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ origin is the fixed point.

orbits

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\cos\varphi \\ 5\sin\varphi \end{pmatrix}$$



All circles of radius r, $r > 0$ are the orbits of this action.

Exercise

$$G' = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\} \leq GL_2(\mathbb{R})$$

G' defines an action on $\mathbb{R}$

$\pi_g(x) = ax+b \qquad G' \times \mathbb{R} \longrightarrow \mathbb{R}$

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}(x) = ax+b$$

A1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(x) = 1 \times x + 0 = x$

A2) $\overset{\pi_g}{\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}} \overset{\pi_f(y)}{\begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}(y)} = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} (\underline{a_2 y + b_2})$

$$= a_1 (\underline{a_2 y + b_2}) + b_1 \in \mathbb{R}$$

$$= \pi_g(\underline{\pi_f(y)}) \qquad \pi_1(x) = x$$

$\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}(0) = a_1 \times 0 + b_1 = b_1$

Discuss the orbit & Stabiliser of 1.

$\pi_g(1) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}(1) = a+b = c$

$\pi g(2) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}(2) = 2a+b = d \in \mathbb{R}$

$G \cdot 1 = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}(1) = a_1 + b_1 \in \mathbb{R}$

$= \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}(1) = a_2 + b_2 \in \mathbb{R}$

orbit of $\{1\}$ is the $\mathbb{R}$.

$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}(1) = 1$

$\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix}(1) = a + 1 - a = 1$

✓

Orbit-counting Lemma

$\pi : G \times X \longrightarrow X$

Theorem: Suppose G is a finite group, and π is an action of G on X. For each $g \in G$, define

$$\text{fix}(g) = \{x \in X \mid \pi_g(x) = x\}.$$

Then the number of orbits of π is

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|.$$

$\pi : G \times X \to X$

$|G| = |\text{orb}(x)|\,|\text{Stab}(x)|$ ✓

Proof: Let N be the number of pairs (g, x) s.t. $\pi_g(x) = x$

First method (I)

$g \in G$, & count x s.t. $\pi_g(x) = x$

$$N = \sum_{g \in G} |\mathrm{fix}(g)|$$

Second method if we take $x \in X$ & count g s.that $\pi_g(x) = x$

$$N = \sum_{x \in X} |\mathrm{Stab}(x)|$$

$$= \sum_{x \in X} \frac{|G|}{|\mathrm{orb}(x)|}$$

Orbit-counting Lemma

$$\sum_{g\in G} |fix(g)| = \sum_{x\in X} \frac{|G|}{|orb(x)|}$$

$$\frac{1}{|G|}\sum_{g\in G} |fix(g)| = \sum_{x\in X} \frac{1}{|orb(x)|}$$

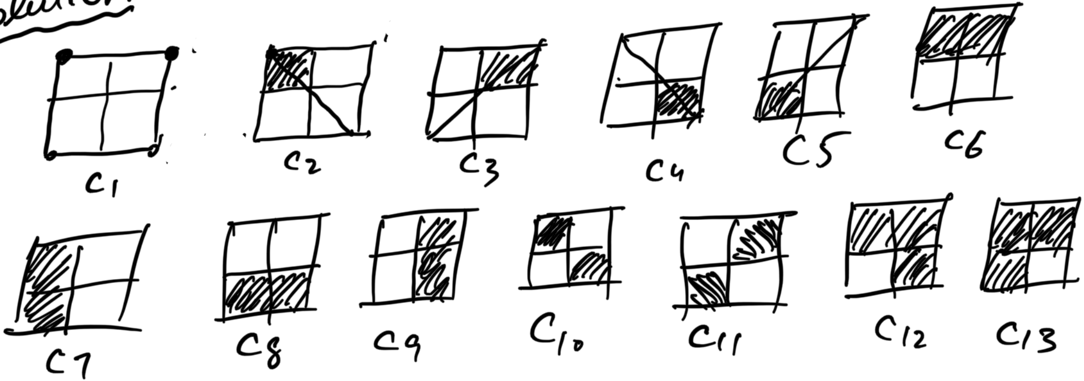
$$\sum_{x\in X} \frac{1}{|orb(x)|} = \sum_{i=1}^{r}\sum_{x\in Y_i} \frac{1}{|Y_i|} = \sum_{i=1}^{r} \frac{|Y_i|}{|Y_i|} = \sum_{i=1}^{r} 1 = r$$

we want to split up the last sum into different orbits $Y_1, Y_2, \dots Y_r$

Orbit-counting Lemma

Example: (The orbit-counting Lemma)
How many different ways are there to color the square using only the black-white.

Solution:



C14

C15

C16

$2^4 = 16$

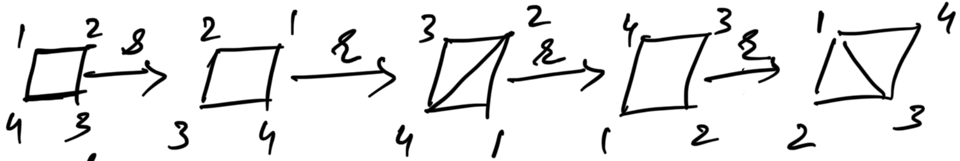
Two colours are equivalent if we apply the symmetry of square.

$$D_8 = \{ e, \overset{90^\circ}{r}, \overset{180^\circ}{r^2}, \overset{270^\circ}{r^3}, s, rs, r^2 s, r^3 s \}$$

$$r \times c_1 = c_1$$

$$O_x \sim O_y \iff x \sim_G y$$

$$O_x = \{ g \cdot x : g \in G \}$$



Elements of $\{x\}$	Orbits	Stabilisers.
c_1	c_1	$D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$
$\{c_2, c_3, c_4, c_5\}$	$\{c_2, c_3, c_4, c_5\}$	$\{r^3s, rs, e\}$
		$Stab_{c_2} = \{e, r^3s\}$
		$Stab_{c_3} = \{e, rs\}$
		$Stab_{c_4} = \{e, r^3s\}$
		$Stab_{c_5} = \{e, rs\}$

c_{10}, c_{11}	$\{c_{10}, c_{11}\}$

x	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}–c_{15}	c_{16}
S_x	8	2	2	2	2	2	2	2	2	4	4	2	8

$$\# \text{ of coloring} = \frac{1}{8}\left(8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8\right)$$

$$= \frac{48}{8} = 6$$

Lemma.

(page 46) Solved Example of orbit counting

Simple Groups

Definition: Suppose G is a group. G is **simple** if $G \neq \{1\}$ and G has no normal subgroups except for G and $\{1\}$.

If G is a group and $\{1\} < N \trianglerighteq G$, then we can break G up into two smaller groups N and G/N, and if we understand these smaller groups then we understand a lot about G. A simple group is one which can't be broken down in this way. This is a bit like prime numbers being the building blocks from which all positive integers are built.

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Examples:

- If p is prime, then the cyclic group $\mathcal{C}_p$ is simple, by Lagrange's Theorem.
- If n is composite, say $n = ab$, then $\mathcal{C}_n$ is not simple: it has a normal subgroup $< z^a >$.
- $\mathcal{D}_{2n}$ is not simple: the subgroup $< r >$ has index 2, so is normal.
- $\mathcal{S}_n$ is not simple if $n \geq 3$, since $\mathcal{A}_n$ is a normal subgroup.

$2 \cdot 3 = 6$

$n = 6$

$C_p = \{1, x, x^2, x^3, x^4\}$

$C_6 = \{1, z, z^2, z^3, z^4, z^5\}$

$z \cdot z^2 \cdot z^5 = z^{3+5} = z^8 = z^2 \in$

$\langle z^2 \rangle = \{z^2, z^4, 1\}$

- D_{2n}, D_8

$$D_8 = \{\underline{1, r, r^2, r^3}, s, rs, r^2s, r^3s\}$$

$$sr = r^{-1}s$$

$$\{1, r, r^2, r^3\}$$

is a normal subgroup of D_8.

$$sr^2s =$$

Simple Groups

Normal

Proposition

Suppose G is an abelian group. Then G is simple if and only if G is finite and $|G|$ is a prime number p, in which case $G \cong C_p$. $\Longleftarrow$

Proof: G is abelian.

$\Longleftarrow$ Suppose G is finite, $|G|$ is prime.

Then by Lagrange's Theorem, the subgroup of G are $\{1\}$ & G, so G is simple.

$G \ni g \neq \{1\}$ ord(g) divides p, $\Rightarrow$ ord$(g) = p$

$G = \{1, g, \dots g^{p-1}\}$

Suppose G is finite, & $|G| > 1$ & is not prime.

Take $g \neq 1 \in G$, $a = \text{ord}(g)$. Then $\langle g \rangle$ is a subgroup of G of order a. If $a < |G|$. Then $\langle g \rangle < G$

so G is not simple. If $a = |G|$,

$a = b \cdot c$, $b, c > 1$ $\text{ord}(g^b) = c$, so

$\{1\} \neq \langle g^b \rangle < G$, G is not simple.

Simple Groups

Simplicity of the alternating groups

(1234)

$= \{id, (12)(34), (13)(24), (14)(23)\}$ $S_4 = 24$

$A_4 = 12$

Consider the subgroup

$$V = \{Id, (12, 34), (13, 24), (14, 23)\}.$$

We saw earlier that V is a normal subgroup of $\mathcal{S}_4$; since it's contained in $\mathcal{S}_4$, it must also be a normal subgroup of $\mathcal{S}_4$. V comprises all the elements of $\mathcal{S}_4$ of cycle type $(1, 1, 1, 1)$ or $(2, 2)$. So the remaining eight elements of $\mathcal{S}_4$ all have cycle type $(3, 1)$. In fact V is the only normal subgroup of $\mathcal{A}_4$ apart from $\{id\}$ and $\mathcal{A}_4$. To prove this, we start by considering actions. Recall that the **natural action** of $\mathcal{S}_n$ on $\{1, \ldots, n\}$ is defined by $\pi_g(x) = g \cdot x$. We can apply this for any subgroup of $\mathcal{S}_n$ as well. Recall also that this action is **transitive** if the only orbit is $\{1, \ldots, n\}$.

$A_4 = \{(123), (134), (124), (132), (143), (142), (234), (243), id, (12)(34), (13)(24), (14)(23)\}$

$V \trianglelefteq A_4, \quad V \trianglelefteq S_4$

Simplicity of the alternating groups

S_n, $N \trianglerighteq A_n$

Lemma

Suppose $\{1\} \neq N \trianglerighteq A_n$. Then the action of N on $\{1, \ldots, n\}$ is transitive.

proof: $id \neq g \in N$, $a \in \{1, 2, \ldots n\}$ $g \cdot a \neq a$

$g \cdot a = b$

$\Rightarrow$ a & b are in same orbit.

$c \in \{1, 2, \ldots n\}$

$h = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

$hgh^{-1} \cdot b = hg \cdot a$

$= h \cdot b = c$ $\Rightarrow$ $c \in orb(a) = orb(a)$

Action of N on $\{1, 2, \ldots n\}$ is transitive.

Simplicity of the alternating groups

Proposition 6.3

The only normal subgroups of $\mathcal{A}_4$ are $\{id\}$, $\mathcal{A}_4$ and V.

Proof: $N \trianglelefteq \mathcal{A}_4$, $N \neq \{id\}$, Action of N on $\{1,2,3,4\}$ is transitive. $|N|$ is divisible by 4.

$|N| \geq 4$, $N \neq V$, Then N contain another element of $\mathcal{A}_4$, with cycle $(3,1)$

$|N|$ is divisible by 3

$|N|$ is divisible by $12 \Rightarrow \mathcal{A}_4 \subseteq N$

Simplicity of the alternating groups

$\{A_{n-1} \leq A_n$

$N \leq A_n$, $N \cap A_n$

$f \cdot n = n$ $\{1, 2, \dots n-1\}$

$\{1, 2, \dots n\}$

Lemma 6.4

Suppose $n \geq 5$ and $\{id\} \neq N \trianglerighteq \mathcal{A}_n$. $\boxed{\text{Then } N \cap \mathcal{A}_{n-1} \neq \{id\}.}$

Proof: The natural action of N on $\{1, 2, \dots n\}$ is transitive, so we can find $g \in N$ s.t. that $\boxed{g \cdot n = 1}$.

If $h \in A_n$, then $ghg^{-1}h^{-1} \in N$. bcs $g \in N$, $hg^{-1}h^{-1} \in N$

by normality & by subgroup.

we can choose $h \in A_{n-1}$ so that

$(ghg^{-1}h^{-1} \in A_{n-1}$ ✓

$g = \begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix}$

$g \cdot n = 1$ ✓

1). $g\cdot 1 = n$, $a \neq 1, n$ s.that $g\cdot a \neq a$

$g = (1\ n)$, Let $b = g\cdot a$, $c \neq 1, a, b, n$

$h = (1\ n)(b\ c) \in A_n$

$ghg^{-1}h^{-1}\cdot n = ghg^{-1}\cdot 1 = gh\cdot n = g\cdot 1 = n$

So $ghg^{-1}h^{-1} \in A_{n-1} \cap N \neq \{id\}$

$ghg^{-1}h^{-1}\cdot c = ghg^{-1}\cdot b = gh\cdot a = g\cdot a = b$

$\boxed{g\cdot n = 1}$

2) $g \cdot 1 \neq n$, let $a = g^{-1} n \Rightarrow ga = n$, $b, c \neq 1$, a, n

$h = \begin{pmatrix} 1 & b & c \\ b & c & 1 \end{pmatrix}$ $g \cdot n = 1$, $\boxed{g^{-1}(1) = n}$

$$ghg^{-1}h^{-1} \cdot n = ghg^{-1} \cdot n = gh\,a = g \cdot a = n$$

$$ghg^{-1}h^{-1} \in A_{n-1} \cap N$$

$$ghg^{-1}h^{-1} \cdot b = ghg^{-1} \cdot n = gh \cdot n = g \cdot n = 1.$$

$$ghg^{-1}h^{-1} \in A_{n-1} \cap N \neq \{id\}$$

Simplicity of the alternating groups

Lemma 6.5

Suppose $N \trianglerighteq \mathcal{A}_5$. Then $N \cap \mathcal{A}_{n-1} \neq V$.

Simplicity of the alternating groups

We give examples to illustrate the above proof. Suppose $N \trianglerighteq \mathcal{A}_6$.

- Suppose $g = (16, 2435) \in N$. We set $h = (16, 45)$. Then $ghg^{-1}h^{-1} = (23, 45) \in N \cap \mathcal{A}_5$.
- Suppose $g = (126, 435) \in N$. We set $h = (134)$. Then $ghg^{-1}h^{-1} = (14253) \in N \cap \mathcal{A}_5$.

$g = (16)(2435) \in N \quad h = (16)(45) \in \mathcal{A}_6$

$= (16)(24)(43)(35)$

$ghg^{-1}h^{-1} = (23)(45) \in N \cap \mathcal{A}_5$

Similarly $\quad g = (126)(435) \in N, \quad h = (134)$

$ghg^{-1}h^{-1} \in N \cap \mathcal{A}_5$

Simplicity of the alternating groups

Lemma 6.6

For $n \geq 5$, $\mathcal{A}_n$ is simple.

Composition Series

Definition: Suppose G is a group. A **normal series** of length r for G is a series

$$G = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r, = \{1\}$$

where $G_0 = G$ and $G_r = \{1\}$.

This series is called a **composition series** if G_i / G_{i+1} is simple for each i

Example:

1) $G = \{1\}$ trivial series

2) G is simple $G \triangleright \{1\}$

composition series of length 1.

Composition Series

Lemma 6.7

Suppose G is a group and $N \trianglelefteq G$. Then G/N is simple if and only if there is no K such that $N < K \trianglelefteq G$.

3) $G = C_{12}$

$$C_{12} \trianglerighteq \langle z^2 \rangle \trianglerighteq \langle z^4 \rangle \trianglerighteq \{1\}$$

$$C_{12} = \{1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}\}$$

$$\langle z^2 \rangle = \{1, z^2, z^4, z^6, z^8, z^{10}\}$$

$\langle g \rangle = \{1, g, g\}$

$$\frac{|C_{12}|}{|\langle z^2 \rangle|} = \frac{12}{6} = 2 \text{ — Simple}$$

$$\frac{|\langle g^2 \rangle|}{|\langle g^4 \rangle|} = \frac{6}{3} = 2 \text{ — Simple}$$

$$\frac{|\langle g^4 \rangle|}{1} = \frac{3}{1} = 3 \text{ — Simp}$$

as these nos are prime.

Composition Series

Corollary 6.8

Every finite group has a composition series.

Consider $D_8 = \{1, r, r^2, r^3, s, rs, r^2 s, r^3 s\}$

$N = \{1, r, r^2, r^3\}$

$N' = \{1, r^2\}$

$D_8 \trianglerighteq N \trianglerighteq N' \trianglerighteq \{1\}$

$\frac{|D_8|}{|N|} = \frac{8}{4} = 2$

$\frac{|N'|}{|N''|} = \frac{4}{2} = 2$

$\frac{|N''|}{1} = 2$ Simple

Composition Series

Jorda-Holder Theorem

Suppose G is a group, and that G has two composition series

$$G_0 \lhd G_1 \lhd \cdots \lhd G_r \qquad \text{and} \qquad H_0 \lhd H_1 \lhd \cdots \lhd H_s \lhd \{1\}.$$

Then $r = s$ and the groups

$$\frac{G_0}{G_1}, \ldots, \frac{G_{r-1}}{G_r}$$

are isomorphic to the groups

$$\frac{H_0}{H_1}, \ldots, \frac{H_{r-1}}{H_r}$$

in some order.

Composition Series

Definition: Suppose G is a group and G has a composition series. The **composition length** of G is the length of any composition series for G, and the **composition factors** of G are the simple groups $G_0/G_1, \ldots, G_{r-1}/G_r$ in any composition series for G.

$S_n : \quad n \geq 5$

$$S_n \trianglerighteq A_n \trianglerighteq \{1\}$$

$$\frac{|S_n|}{|A_n|} = \frac{2n!}{2n!/n} = 2 \leftarrow \text{simple}$$

Composition Series

Example:

- If $G = \{1\}$, then G has no composition factors.
- If G is simple, then the only composition factor of G is G.
- If $G = \mathcal{C}_{12}$, then we saw that the quotients in a composition series for G have orders $2, 2, 3$. Since any group of order p (for p a prime) is isomorphic to $\mathcal{C}_p$, the composition factors of $\mathcal{C}_{12}$ are $\mathcal{C}_2, \mathcal{C}_2, \mathcal{C}_3$. (Note that when we list the composition factors of a group, factors can appear more than once.)
- If $G = \mathcal{S}_n$ for $n \geq 5$, then the composition factors of G are $\mathcal{C}_2$ and $\mathcal{A}_n$.

Exams Style Questions

Example: Find the composition series of S_4.

$$S_4 \trianglerighteq A_4 \trianglerighteq V \trianglerighteq \{1\}$$

$$\frac{|S_4|}{|A_4|} = 2, \quad \frac{|A_4|}{|V|} = \frac{12}{4} = 3 \quad \text{simple}$$

Exams Style Questions

Example: Find the composition series of

C_{27}

Solution: $C_{27} \trianglerighteq C_9 \trianglerighteq C_3 \trianglerighteq \{1\}$

$C_9 = \langle x^3 \rangle$

$C_3 = \langle x^9 \rangle$

$$\frac{|C_{27}|}{9} = \frac{27}{9} = 3$$

$$\frac{\langle x^3 \rangle}{\langle x^9 \rangle} = 3$$

Example: Find two different composition series of D_{12}.

Solution: $D_{12} \trianglerighteq C_6 \trianglerighteq C_3 \trianglerighteq \{1\}$

$C_6 = \{1, r, r^2, r^3, r^4, r^5\}$

$\langle r^2 \rangle = \{1, r^2, r^4\} \cong C_3$

$\langle r^3 \rangle = \{1, r^3\} \cong C_2$

$\langle s \rangle = \{1, s\}$

$D_{12} \trianglerighteq C_2 \trianglerighteq \{1\}$

QMplus Quiz

Attempt Quiz 10 at QMplus page

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$