1. $\left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right] u(\partial x, \partial^2 t)$

$= \frac{\partial}{\partial t}\left[u(\partial x, \partial^2 t)\right] - k\frac{\partial^2}{\partial x^2}\left[u(\partial x, \partial^2 t)\right]$

$= \partial^2 \cdot u_t(\partial x, \partial^2 t) - k \cdot \partial^2 u_{xx}(\partial x, \partial^2 t)$

$= \partial^2 \cdot \left[u_t(\partial x, \partial^2 t) - k u_{xx}(\partial x, \partial^2 t)\right]$

$= \partial^2 \cdot 0$ $\Leftarrow$ using u satisfies heat equation $u_t - ku_{xx} = 0$

$= 0$

So $u(\partial x, \partial^2 t)$ satisfies Heat Equation

On the other hand, $u(\partial x, -\partial^2 t)$ does not satisfies the Heat Equation

2. $\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(e^{ax+bt})$

$= b \cdot e^{ax+bt} - k \cdot a^2 e^{ax+bt}$

$= (b - ka^2)e^{ax+bt}$

so any (a, b) satisfying $b = ka^2$ will make e^{ax+bt} a solution to heat equation.

3

By Fourier-Poisson formula.

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy \quad \text{with } f \equiv 1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x-y|^2}{4kt}} \cdot 1\, dy$$

change of variables $\tilde{y} = x-y$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\tilde{y}^2}{4kt}} d\tilde{y}$$

$$= 1$$

the temperature for all later time is 1, the same as initial temperature.

4

By F-P formula.

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy$$

By the definition of $f(y)$.

$$= \int_{-L}^{L} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{|x-y|^2}{4kt}} \cdot 1 \cdot dy$$

Doing a change of variable

$$s = \frac{x-y}{\sqrt{4kt}}, \qquad -\sqrt{4kt}\cdot ds = dy$$

we get

$$u(x,t) = \int_{\frac{x+L}{\sqrt{4kt}}}^{\frac{x-L}{\sqrt{4kt}}} \frac{1}{\sqrt{4\pi kt}} e^{-s^2} (-\sqrt{4kt})\, ds$$

$$= \int_{\frac{x-L}{\sqrt{4kt}}}^{\frac{x+L}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

so $\lim_{t\to\infty} u(x,t) = \int_{\lim_{t\to\infty}\frac{x-L}{\sqrt{4kt}}}^{\lim_{t\to\infty}\frac{x+L}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds$

$$= \int_0^0 \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

$$= 0$$

6.

By F-P formula,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} e^{3y} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt} + 3y} dy$$

Notice $-\frac{(x-y)^2}{4kt} + 3y = -\frac{x^2 - 2xy + y^2 - 12kty}{4kt}$

$$= -\frac{(y-x-6kt)^2}{4kt} + 9kt + 3x$$

So $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(y-x-6kt)^2}{4kt} + 9kt + 3x} dy$

$$= e^{9kt+3x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(y-x-6kt)^2}{4kt}} dy$$

Do change of variables

$$s = \frac{y-x-6k}{\sqrt{4kt}}, \quad \sqrt{4kt}\, ds = dy$$

We get

$$u(x,t) = e^{9kt+3x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-s^2} \cdot \sqrt{4kt}\, ds$$

$$= e^{9kt+3x} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds \right]$$

Notice $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$

$$= e^{9kt+3x} \left[\frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \right]$$

$$= e^{9kt+3x} \to +\infty$$

as $t \to \infty$