# Group Theory

**Week 10, Lecture** 1, 2 & 3

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# Group Actions

Suppose $G$ is a group and $X$ is a set. An action of $G$ on $X$ is a collection $\pi = (\pi_g \,|g \in G)$ of functions from $X$ to $X$ such that:

1. $\pi_1 = \mathrm{id}_X$, and
2. $\pi_f \circ \pi_g = \pi_{fg}$ for all $f, g \in G$.

# Group Actions

**Exercise.** Let $G = GL(2, \mathbb{R})$ and $X = \mathbb{R}^2$.

(1) Show that the map

$$G \times X \to X, \qquad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

defines a G action.

(2) What are the orbits and fixed point sets of this G action?

The collection of all invertible matrices constitutes the general linear group $GL(2, \mathbb{R})$.

# Group Actions

**Exercise.** Let $H \leq G$, and define $H$ action by restricting the map $H \times X$. Calculate the orbits and fixed point sets in the following cases:

(1) $H = SO(2)$.

(2) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(3) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(4) $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$.

(5) $H = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$.

# Group Actions

# Orbit-counting Lemma

**Theorem**: Suppose $G$ is a finite group, and $\pi$ is an action of $G$ on $X$. For each $g \in G$, define

$$\text{fix}(g) = \{x \in X \mid \pi_g(x) = x\}.$$

Then the number of orbits of $\pi$ is

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|.$$

# Orbit-counting Lemma

# Orbit-counting Lemma

**Example**:

# Simple Groups

**Definition**: Suppose $G$ is a group. $G$ is **simple** if $G \neq \{1\}$ and $G$ has no normal subgroups except for $G$ and $\{1\}$.

If $G$ is a group and $\{1\} < N \trianglerighteq G$, then we can break $G$ up into two smaller groups $N$ and $G/N$, and if we understand these smaller groups then we understand a lot about $G$. A simple group is one which can't be broken down in this way. This is a bit like prime numbers being the building blocks from which all positive integers are built.

**Examples**:

- If $p$ is prime, then the cyclic group $\mathcal{C}_p$ is simple, by Lagrange's Theorem.
- If $n$ is composite, say $n = ab$, then $\mathcal{C}_n$ is not simple: it has a normal subgroup $< z^a >$.
- $\mathcal{D}_{2n}$ is not simple: the subgroup $< r >$ has index 2, so is normal.
- $\mathcal{S}_n$ is not simple if $n \geq 3$, since $\mathcal{A}_n$ is a normal subgroup.

# Simple Groups

## Proposition

Suppose $G$ is an abelian group. Then $G$ is simple if and only if $G$ is finite and $|G|$ is a prime number $p$, in which case $G \cong \mathcal{C}_p$.

# Simple Groups

# Simple Groups

# Simplicity of the alternating groups

Consider the subgroup

$$V = \{Id, (12, 34), (13, 24), (14, 23)\}.$$

We saw earlier that $V$ is a normal subgroup of $\mathcal{S}_4$; since it's contained in $\mathcal{S}_4$, it must also be a normal subgroup of $\mathcal{S}_4$. $V$ comprises all the elements of $\mathcal{S}_4$ of cycle type $(1, 1, 1, 1)$ or $(2, 2)$. So the remaining eight elements of $\mathcal{S}_4$ all have cycle type $(3, 1)$. In fact $V$ is the only normal subgroup of $\mathcal{A}_4$ apart from $\{id\}$ and $\mathcal{A}_4$. To prove this, we start by considering actions. Recall that the **natural action** of $\mathcal{S}_n$ on $\{1, \ldots, n\}$ is defined by $\pi_g(x) = g \cdot x$. We can apply this for any subgroup of $\mathcal{S}_n$ as well. Recall also that this action is **transitive** if the only orbit is $\{1, \ldots, n\}$.

# Simplicity of the alternating groups

**Lemma**

Suppose $\{1\} \neq N \trianglerighteq \mathcal{A}_n$. Then the action of $N$ on $\{1, \ldots, n\}$ is transitive.

## Simplicity of the alternating groups

**Proposition 6.3**

The only normal subgroups of $\mathcal{A}_4$ are $\{id\}$, $\mathcal{A}_4$ and $V$.

# Simplicity of the alternating groups

**Lemma 6.4**

Suppose $n \geq 5$ and $\{id\} \neq N \trianglerighteq \mathcal{A}_n$. Then $N \cap \mathcal{A}_{n-1} \neq \{id\}$.

# Simplicity of the alternating groups

**Lemma 6.5**

Suppose $N \trianglerighteq \mathcal{A}_5$. Then $N \cap \mathcal{A}_{n-1} \neq V$.

# Simplicity of the alternating groups

We give examples to illustrate the above proof. Suppose $N \trianglerighteq \mathcal{A}_6$.

- Suppose $g = (16, 2435) \in N$. We set $h = (16, 45)$. Then $ghg^{-1}h^{-1} = (23, 45) \in N \cap \mathcal{A}_5$.
- Suppose $g = (126, 435) \in N$. We set $h = (134)$. Then $ghg^{-1}h^{-1} = (14253) \in N \cap \mathcal{A}_5$.

# Simplicity of the alternating groups

**Lemma 6.6**

For $n \geq 5$, $\mathcal{A}_n$ is simple.

# Composition Series

**Definition**: Suppose $G$ is a group. A **normal series** of length $r$ for $G$ is a series

$$G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_r,$$

where $G_0 = G$ and $G_r = \{1\}$.
This series is called a **composition series** if $G_i/G_{i+1}$ is simple for each $i$

**Example**:

# Composition Series

**Lemma 6.7**

Suppose $G$ is a group and $N \trianglelefteq G$. Then $G/N$ is simple if and only if there is no $K$ such that $N < K \trianglelefteq G$.

# Composition Series

**Corollary 6.8**

Every finite group has a composition series.

# Composition Series

## Jorda-Holder Theorem

Suppose $G$ is a group, and that $G$ has two composition series

$$G_0 \lhd G_1 \lhd \cdots \lhd G_r \qquad \text{and} \qquad H_0 \lhd H_1 \lhd \cdots \lhd H_s \lhd \{1\}.$$

Then $r = s$ and the groups

$$\frac{G_0}{G_1}, \ldots, \frac{G_{r-1}}{G_r}$$

are isomorphic to the groups

$$\frac{H_0}{H_1}, \ldots, \frac{H_{r-1}}{H_r}$$

in some order.

# Composition Series

**Definition**: Suppose $G$ is a group and $G$ has a composition series. The **composition length** of $G$ is the length of any composition series for $G$, and the **composition factors** of $G$ are the simple groups $G_0/G_1, \ldots, G_{r-1}/G_r$ in any composition series for $G$.

# Composition Series

**Example**:

- If $G = \{1\}$, then $G$ has no composition factors.
- If $G$ is simple, then the only composition factor of $G$ is $G$.
- If $G = \mathcal{C}_{12}$, then we saw that the quotients in a composition series for $G$ have orders $2, 2, 3$. Since any group of order $p$ (for $p$ a prime) is isomorphic to $\mathcal{C}_p$, the composition factors of $\mathcal{C}_{12}$ are $\mathcal{C}_2, \mathcal{C}_2, \mathcal{C}_3$. (Note that when we list the composition factors of a group, factors can appear more than once.)
- If $G = \mathcal{S}_n$ for $n \geq 5$, then the composition factors of $G$ are $\mathcal{C}_2$ and $\mathcal{A}_n$.

# Exams Style Questions

# Exams Style Questions

# QMplus Quiz

**Attempt Quiz 10 at QMplus page**

## Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order $n$.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and $n$ which are prime to $n$, with the group operation being multiplication modulo $n$.

## Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$