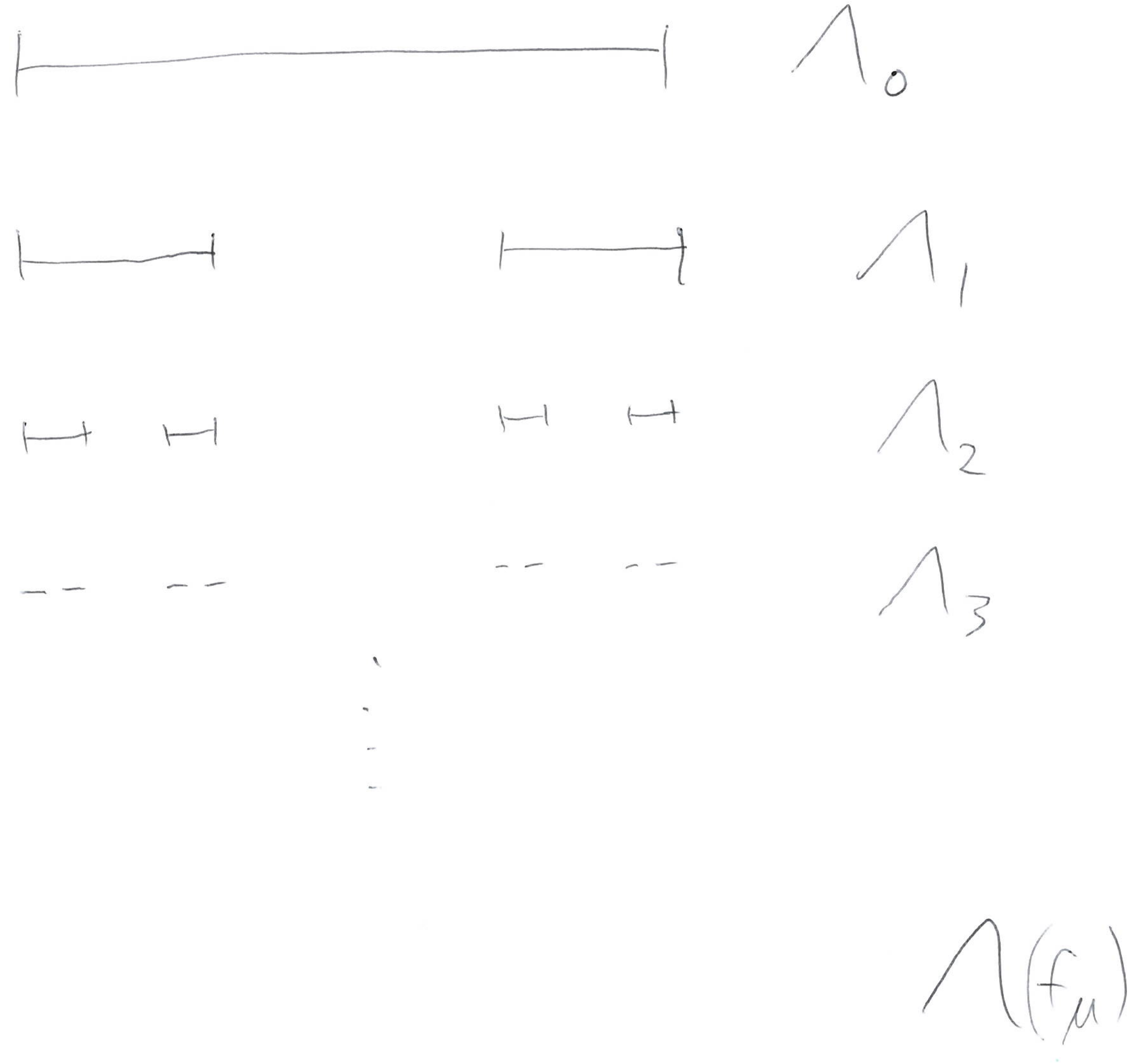
The non-escaping set $\Lambda(f_\mu)$ can be written as

$$\Lambda(f_\mu) = \bigcap_{n=0}^{\infty} \Lambda_n$$

and this is a Cantor set



The points in $\Lambda = \Lambda(f_\mu)$ correspond naturally to the set S of sequences whose entries are 0 or 1.

If $b = b_1 b_2 b_3 b_4 \ldots$ is such a sequence then there is a (unique) point $x_0 \in \Lambda(f_\mu)$ such that

$$f_\mu^{i-1}(x_0) = x_{i-1} \in I_{b_i} \quad \forall i \in \mathbb{N}$$

In fact:

<u>Proposition</u>: For $\mu > 4$, the non-escaping set $\Lambda(f_\mu)$ is a Cantor set, and the dynamical system $f_\mu : \Lambda(f_\mu) \to \Lambda(f_\mu)$ is (topologically) conjugate to the shift map $\sigma : S \to S$.

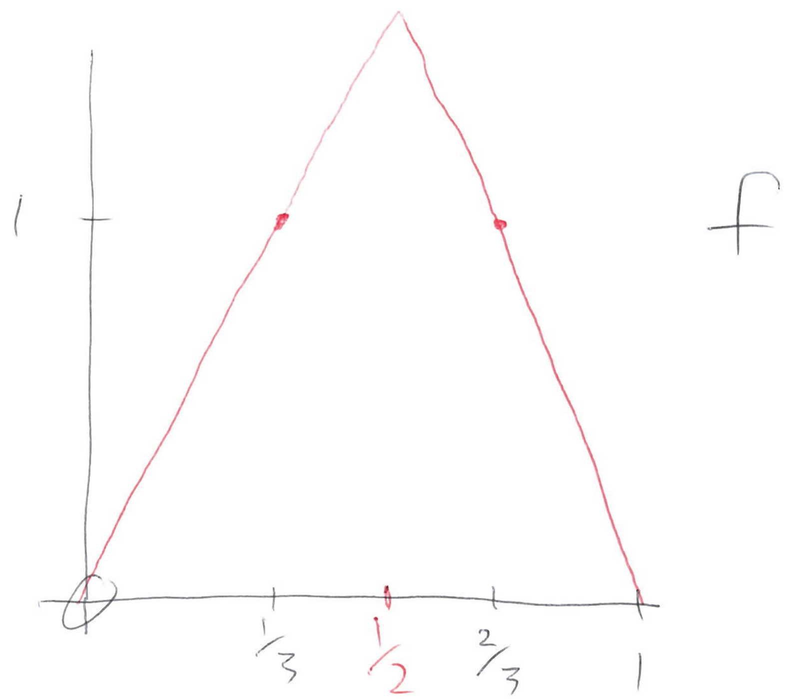
The diagram

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "h"'] & S \arrow[d, "h"] \\ \Lambda(f_\mu) \arrow[r, "f_\mu"] & \Lambda(f_\mu) \end{tikzcd}$$

Commutes, i.e. $h \circ \sigma = f_\mu \circ h$,
where $h(b_1 b_2 b_3 \ldots)$ is defined to be the point $x_0 \in \Lambda(f_\mu)$ such that $x_{i-1} \in I_{b_i}$ for all $i \in \mathbb{N}$.

<u>Corollary</u>

Example Viewing the Middle-Third Cantor set as a non-escaping set



Define $f:[0,1]\to\mathbb{R}$ by

$$f(x)=\begin{cases}3x & \text{if } x\in[0,1/2)\\ 3-3x & \text{if } x\in[1/2,1]\end{cases}$$

Then the non-escaping set

$$\Lambda(f)=\{x\in[0,1]:f^n(x)\in[0,1]\ \forall n\geq 0\}$$

is <u>precisely</u> the middle-third Cantor set.

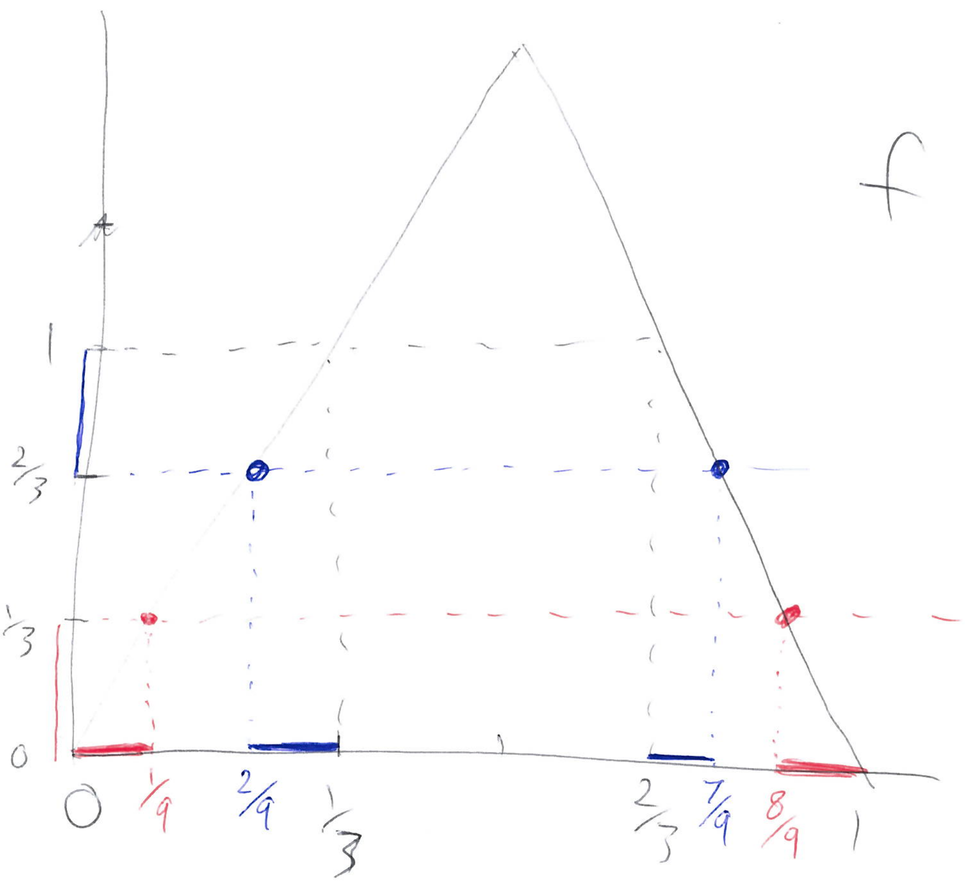
To see this, note that

$$\Lambda_1 = \{ x \in [0,1] : f(x) \in [0,1] \}$$

$$= f^{-1}([0,1])$$

$$= [0, 1/3] \cup [2/3, 1]$$

$$\Lambda_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$



A related viewpoint

Alternatively, we could introduce maps $\varphi_1 : [0,1] \to [0,1]$ and

$$\varphi_2 : [0,1] \to [0,1]$$

defined by $\varphi_1(x) = x/3$

and $\varphi_2(x) = x/3 + 2/3 = \dfrac{x+2}{3}$

Note that $\varphi_1([0,1]) = [0, 1/3]$

and $\varphi_2([0,1]) = [2/3, 1]$

Now define a mapping Φ to act on subsets of $[0,1]$ by the formula

$$\Phi(A) := \varphi_1(A) \cup \varphi_2(A)$$

(where $\varphi_i(A) = \{\varphi_i(a) : a \in A\}$ for $i = 1, 2$)

Then $\Phi([0,1])$

$$= \varphi_1([0,1]) \cup \varphi_2([0,1])$$

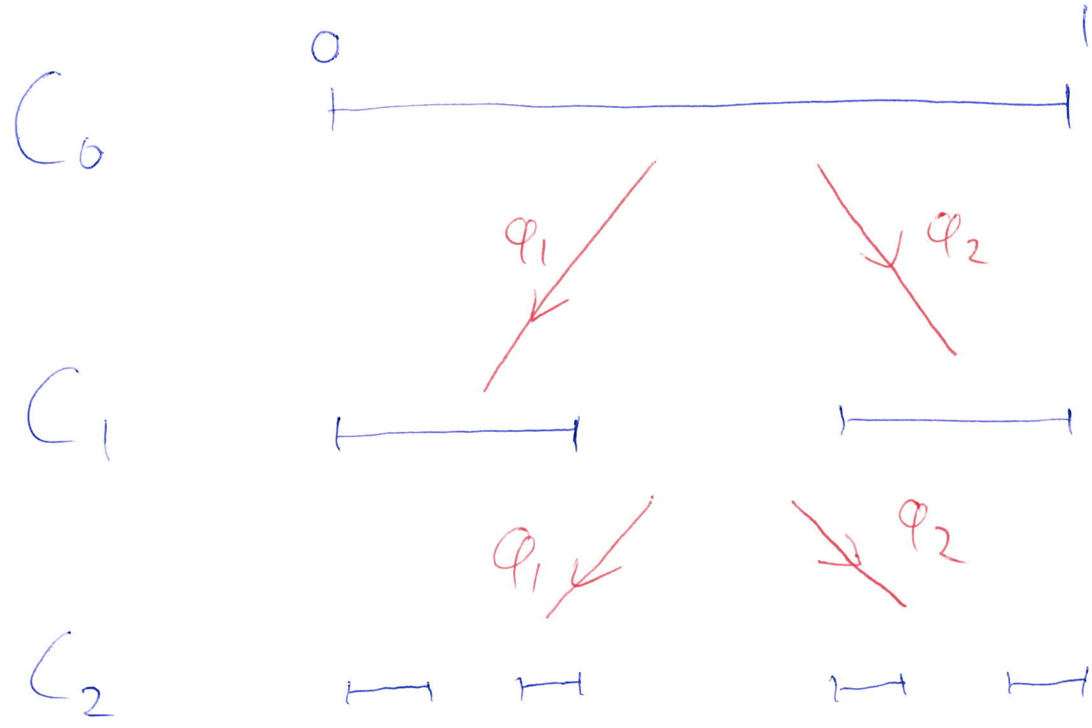
$$= [0, 1/3] \cup [2/3, 1]$$

Also $\Phi^2([0,1]) = \Phi(\Phi([0,1]))$

$$= \Phi\left([0, 1/3] \cup [2/3, 1]\right)$$

$$= \varphi_1([0,1/3] \cup [2/3,1]) \cup \varphi_2([0,1/3] \cup [2/3,1])$$

$$= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$



In fact $\Phi^n([0,1])$ is the set C_n in the definition of the middle-third Cantor set C.

So $C = \bigcap_{n=0}^{\infty} C_n = \bigcap_{n=0}^{\infty} \Phi^n([0,1])$

<u>Remark</u> We can think of C as a fixed point of Φ, i.e. $\Phi(C) = C$. There is also a sense in which $\Phi^n([0,1])$ <u>converges</u> to C as $n \to \infty$ (this is a more advanced viewpoint).

This viewpoint suggests the following:

<u>Defn</u> A finite collection of maps $\varphi_1, \dots, \varphi_k$ where each $\varphi_i : [0,1] \to [0,1]$ is called an <u>iterated function system</u>

We define an associated map Φ, acting on subsets of $[0,1]$, by

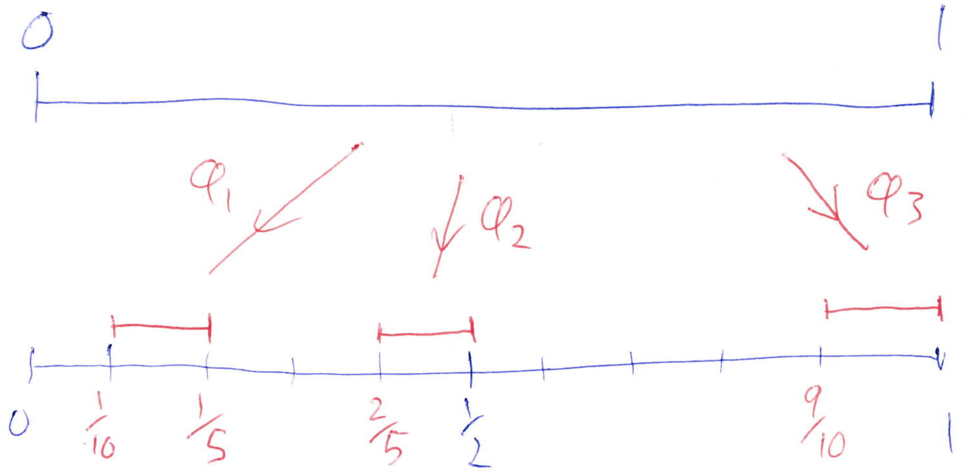
$$\Phi(A) = \bigcup_{i=1}^{k} \varphi_i(A)$$

<u>Example</u> Define

$$\varphi_1(x) = \frac{x+1}{10}$$

$$\varphi_2(x) = \frac{x+4}{10}$$

$$\varphi_3(x) = \frac{x+9}{10}$$



Note that $\varphi_1([0,1]) = [1/10, 2/10]$

$= \{\text{numbers with decimal expansion } 0\cdot 1\ldots\}$

$\varphi_2([0,1]) = [4/10, 5/10]$

$= \{\text{numbers with decimal expansion } 0\cdot 4\ldots\}$

$\varphi_3([0,1]) = [9/10, 1]$

$= \{\text{numbers with decimal expansion } 0\cdot 9\ldots\}$

So $\Phi([0,1]) = [1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]$

Then $\Phi^2([0,1]) = \Phi(\Phi([0,1]))$

$= \Phi\left([1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]\right)$

$= \varphi_1\left([1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]\right)$

$\cup\ \varphi_2\left(\quad " \quad\quad " \quad\quad " \quad\quad " \quad\quad " \quad\right)$

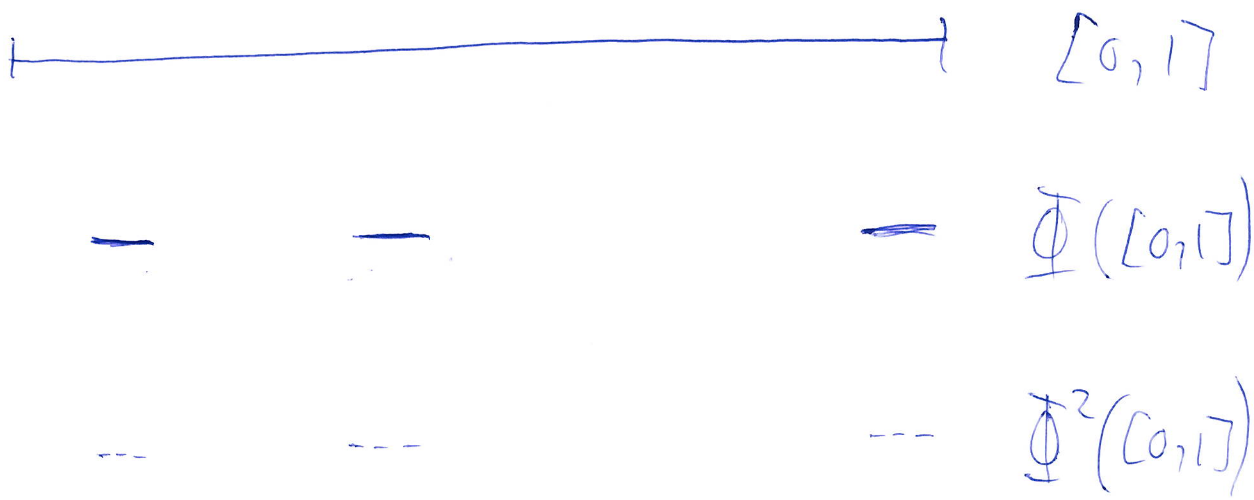
$\cup\ \varphi_3\left(\quad " \quad\quad " \quad\quad " \quad\quad " \quad\quad " \quad\right)$

$$= \left[\frac{11}{100}, \frac{12}{100}\right] \cup \left[\frac{14}{100}, \frac{15}{100}\right] \cup \left[\frac{19}{100}, \frac{20}{100}\right]$$

$$\cup \left[\frac{41}{100}, \frac{42}{100}\right] \cup \left[\frac{44}{100}, \frac{45}{100}\right] \cup \left[\frac{49}{100}, \frac{50}{100}\right]$$

$$\cup \left[\frac{91}{100}, \frac{92}{100}\right] \cup \left[\frac{94}{100}, \frac{95}{100}\right] \cup \left[\frac{99}{100}, 1\right]$$

(Note that e.g. $\left[\frac{14}{100}, \frac{15}{100}\right]$ is precisely the set of all numbers which admit a decimal expansion whose first 2 digits are 14, i.e. they have decimal expansion $0.14\ldots$)



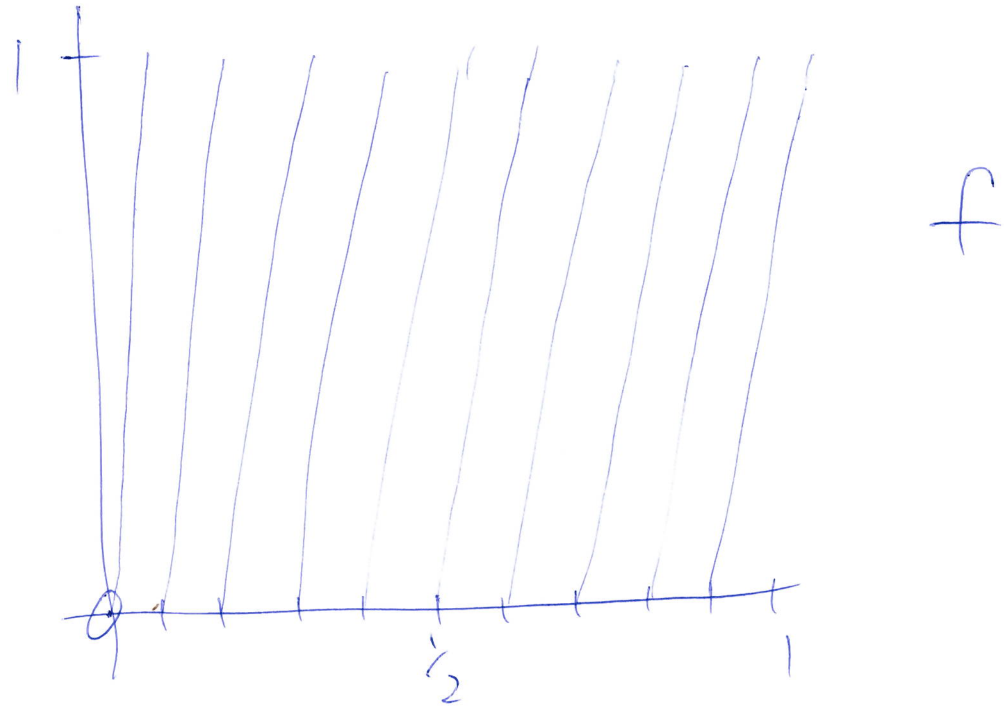
The intersection $\bigcap_{n=0}^{\infty} \Phi^n([0,1])$ is a Cantor set, let's call it K. It is the set of all numbers in $[0,1]$ which have a decimal expansion only containing the digits 1, 4 and 9.

Also $\Phi(K) = K$.

Also, if we define $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 10x \pmod 1 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

(i.e. $f(x) = 10x - i$ if $x \in [i/10, \frac{i+1}{10}]$ for $0 \le i \le 9$)



Then $f(K) = K$

In view of the fact that $f(K) = K$ we might consider the restricted map $f|_K : K \to K$. This restricted map is topologically conjugate to the shift map σ on the set of sequences on a 3-digit alphabet $\{1,4,9\}$

It is natural to extend the definition of iterated function system to higher dimensions:

<u>Defn</u> A finite collection $\varphi_1, \dots, \varphi_k$ of map $\varphi_i : \mathbb{R}^d \to \mathbb{R}^d$ is called an <u>iterated function system</u> (on $\mathbb{R}^d$).

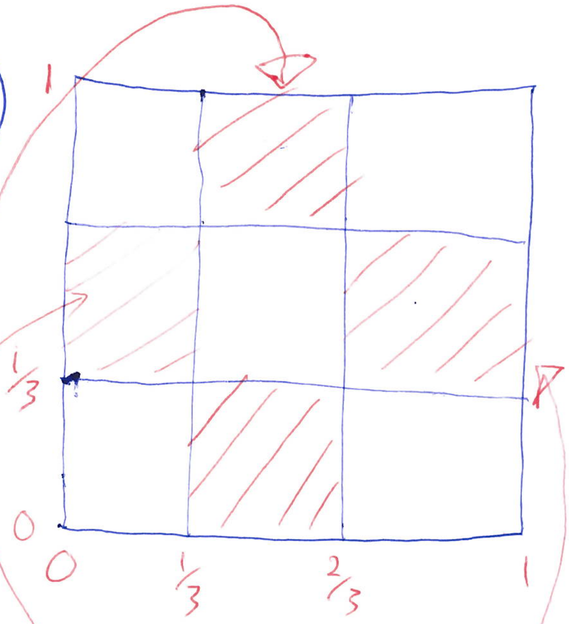
The associated map Φ acting on subsets of $\mathbb{R}^d$ is defined by

$$\Phi(A) = \bigcup_{i=1}^{k} \varphi_i(A)$$

Example $d=2$, $K=4$, $\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi_1(x,y) := \left(\frac{x+1}{3}, \frac{y}{3}\right)$$

$$\varphi_1([0,1]^2) = [1/3, 2/3] \times [0, 1/3]$$



$$\varphi_2(x,y) = \left(\frac{x}{3}, \frac{y+1}{3}\right)$$

$$\varphi_2([0,1]^2) = [0, 1/3] \times [1/3, 2/3]$$

$$\varphi_3(x,y) = \left(\frac{x+2}{3}, \frac{y+1}{3}\right)$$

$$\varphi_3([0,1]^2) = [2/3, 1] \times [1/3, 2/3]$$

$$\varphi_4(x,y) = \left(\frac{x+1}{3}, \frac{y+2}{3}\right)$$

$$\varphi_4([0,1]^2) = [1/3, 2/3] \times [2/3, 1]$$

As usual, define $\Phi(A) = \bigcup_{i=1}^{4} \varphi_i(A)$

Let $F_0 = [0,1]^2$

$$F_1 = \Phi([0,1]^2) = \Phi(F_0)$$

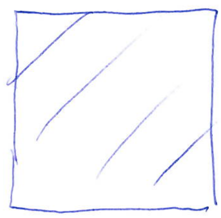
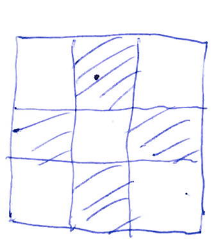
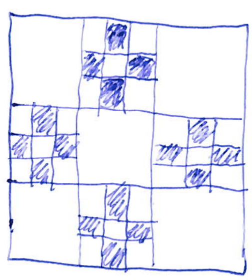
In general $F_n = \Phi^n([0,1]^2)$

$$= \Phi^n(F_0)$$

Let $F = \bigcap_{n=0}^{\infty} F_n = \bigcap_{n=0}^{\infty} \Phi^n([0,1]^2)$

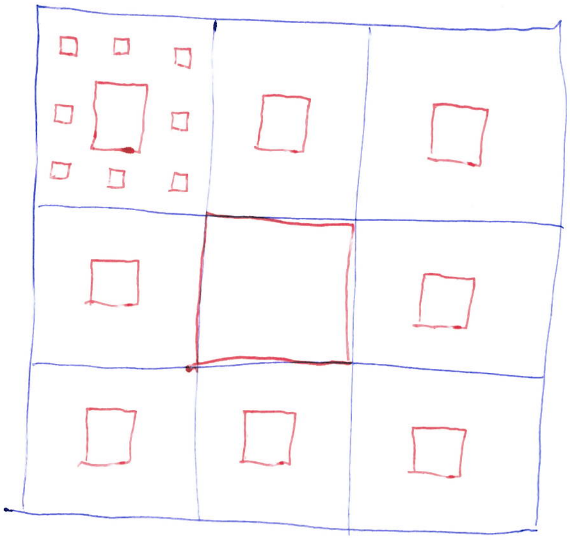
This is a Cantor set.

Pictorially:

n	F_n
0	
1	
2	

More examples

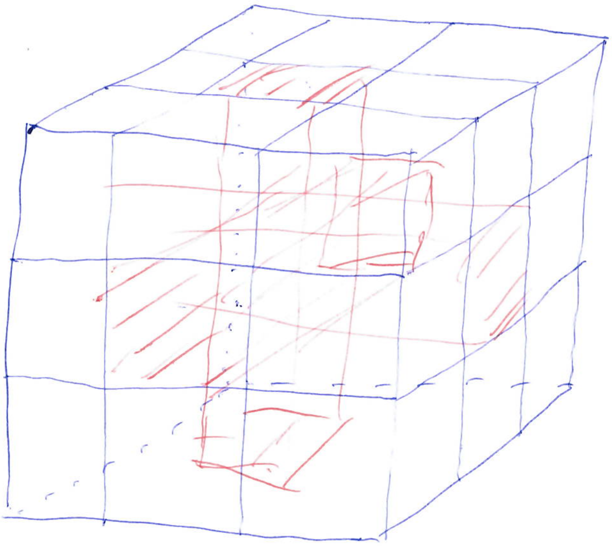
Sierpinski Carpet



General rule:

- Divide a square into 9
- Remove the central square

Menger Sponge



General rule:

- Divide a cube into 27 smaller cubes (each of size $\frac{1}{3}$
- Remove the "central rectanguloid" from all 3 directions