Maths & Stats Pre-Sessional Tutorial

Topic 4: Simple Functions

Exercise 1

A large company would like to investigate if the level of education, measured in years of education (EDU), is determinant in having a higher salary. Using the company's human resource information, the following model has been estimated:

$$\ln(\text{WAGE}) = 1.439 + .0834\ \text{EDU}$$

a) Define which variable in the above model is the dependent and which one is the independent variable.

Solution: Independent variables are what we expect will influence dependent variables. A dependent variable is what happens as a result of the independent variable. In this example, the dependent variable is ln(wage), and the independent variable is EDU.

b) Is 0.0834 an estimator or an estimate? Explain your answer.

Solution: 0.0834 is an estimate. an estimator is a rule for calculating an estimate of a given quantity based on observed data. The numerical value obtained by the use of an estimator is an estimate.

c) You want to test whether having more years of education expresses the possibility of having a higher salary. Write down the null and the alternative hypotheses.

Solution: We can wite this model in a more generic way

$$\ln{(wage)_i} = \alpha + \beta \text{EDU}_i + \text{u}_i$$

The null and the alternative hypotheses are:

$$H_0: \beta = 0$$
$$H_0: \beta > 0$$

d) Determine whether the variable years of education has a positive or negative impact on the salary.

Solution: Since the slope is positive (+0.0834), we can conclude the variable years of education has a positive impact.

e) Is the above model a linear model?

Solution: yes, it is a linear model. Linear model refers to any model which assumes linearity in the system.

f) Draw the regression line.

Let's define two points through which the line passes:

If x is zero, then the line passes through (0, 1.439)

If x is 1, then the line passes through (1, 1.5224)

g) If years of education does not have any impact on the salary, how the regression line changes. Draw the new line.

Solution: If years of education does not have any impact on the salary, then the model will be a constant line. It will be an horizontal line that goes through the point (0, 1.439)

Exercise 2

Select the correct answer:

The ratio of the change in y-values to the change in x-values is called:

a) Dependent variable
b) Independent variable
c) Intercept
d) Slope

Explanation: The slope of a line is a measure of its steepness. The slope of a line is the ratio of the 'change' in y to the 'change' in x (ratio of vertical change to horizontal change)

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$$

Exercise 3

Select the correct answer:

A horizontal line has a slope with a value that is:

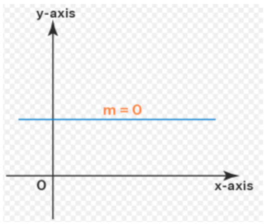
a) positive

b) undefined

c) negative

d) zero

Explanation: A horizontal line has slope zero since it does not rise vertically while a vertical line has undefined slope since it does not run horizontally



Exercise 4

Select the correct answer:

Standard form for a linear equation is:

a) y = x

b) y = mx + b

c) Y-Y1 / X-X1

d) Ax + By = C

Exercise 5

Write down the functions and draw the graph of the functions:

a) passing through (0,1) and having slope -2

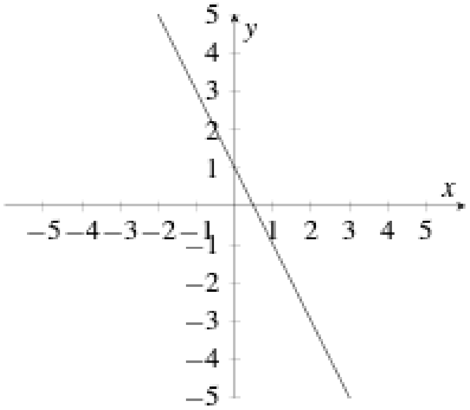
b) passing through (-2,2) and parallel to: y = 2 — 5x

Solution:

(a) A linear function can be written in the general form as $y = a + bx$. By substituting the value of the slope and point coordinates given for b and x and y respectively, we obtain:

$$\begin{cases} y = a - 2x \\ 1 = a \end{cases}$$

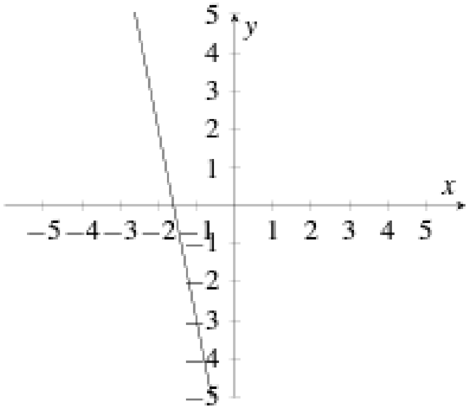
which in turn implies: $y = 1 - 2x$.



(b) Since the function must be parallel to the one give, the slope of the new function must be the same that is, $b = -5$. Using this information and the point coordinates, we obtain:

$$\begin{aligned} y &= a + bx \\ 2 &= a - 2b \\ 2 &= a + 10 \\ a &= -8 \end{aligned}$$

which implies: $y = -8 - 5x$.



Exercise 6

Consider the following quadratic function:

i. $f(x) = -x^2 + 8x - 12$
ii. $f(x) = 2x^2 - 12x - 3$
iii. $f(x) = x^2 + 6x + 9$

a) Explain whether the functions are convex or concave.

Solution: A quadratic function in a form $f(x) = ax^2 + bx + c$ is convex if a ≥ 0 and concave if a ≤ 0. Therefore: i) is concave, ii) is convex and iii) is convex.

b) After solving an optimisation problem for equation (ii), we obtain the following point (3, -21). Is this point a maximum or a minimum point of the function?

Solution: This point is a minimum of the function. Using the quadratic formula, we can show it graphically.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 * 2 * (-3)}}{2 * 2} = \frac{12 \pm 13}{2 * 2}$$

We have two possible solutions:

$x = \frac{12+13}{2*2} =$ 6.25 and $x = \frac{12-13}{2*2} = -0.25$

The curve passes through the points (-0.25, 0) and (6.25,0)

Exercise 7

Compute simple return and log return for the stock below and complete the table.

Period	Price	Simple Return	Log Return
Year 1	100	-	-
Year 2	200		
Year 3	100		
	Average return:		

Solution:

Period	Price	Simple Return	Log Return
Year 1	100	-	-
Year 2	200	100%	69.31%
Year 3	100	−50%	−69.31%
	Average return:	25%	0%

Simple return for year 2 is calculated as $\frac{P_2-P_1}{P_1} = \frac{200-100}{100}$

Log return for year 2 is calculated as ln(200) – ln(100) or as ln(200/100)

Similarly, it can be computed for year 3.

Exercise 8

An economy is forecast to grow continuously so that the gross national product (GNP) measured in billions of dollars, after t years is given by:

$$88 = 80e^{0.02t}$$

After how many years is GNP forecast to be $88 billion? What does the model predict about the value of GNP in the long run?

Solution: we need to solve

$$88 = 80e^{0.02t}$$

$$\frac{88}{80} = e^{0.02t}$$

$$1.1 = e^{0.02t}$$

$$\ln(1.1) = 0.02t\ ln(e)$$

$$\ln(1.1) = 0.02t$$

$$t = \frac{\ln(1.1)}{0.02} = \frac{0.09531}{0.02} = 4.77$$

We therefore deduce that GNP reaches a level of $88 billion after 4.77 years.

A graph of GNP plotted against time (this is an exponential function) would show that GNP just keeps on rising over time.

Exercise 9

A model for GDP, g, measured in billions of dollars, over a period of t years can be formulated in the form : $g = Be^{At}$, where A and B are two parameters. Can this model be estimated using a linear regression model? Explain your answer.

Solution: The basic shape of the curve joining these points suggests that an exponential function is likely to provide a reasonable model. However, since one of the unknown parameters, A, occurs as a power in the relation, it is a good idea to take natural logs of both sides to get:

$$\ln(g) = \ln(Be^{At})$$

$$\ln(g) = \ln(B) + \ln(e^{At})$$

$$\ln(g) = \ln(B) + At\ln(e)$$

$$\ln(g) = \ln(B) + \mathrm{At}$$

Although this does not look like it at first sight, this relationship is actually the equation of a straight line. We can recall:

$$\mathrm{y} = \ln(g) \ \ and\ x = t$$

This equation becomes

$$\mathrm{y} = \ln(B) + \mathrm{Ax}$$

So, a graph of ln(g) plotted on the vertical axis with t plotted on the horizontal axis should produce a straight line with slope A and with intercept of ln(B).

Exercise 10

Compute the present value of £500 to be received in one year's time given the interest rate of 8%.

Solution:

The present value PV of an amount V to be received t periods from now, with interest rate r per period and compounding that occurs at the end of each period is:

$$PV_t = \frac{V}{(1+r)^t}$$

Since t = 1, we have:

$$PV_1 = \frac{V}{(1+r)^1} = \frac{500}{1+0.08} = 462.96$$

Exercise 11

Compute the present value of receiving 1 million at the end of each of the next three years given the interest rate of 12%.

Solution:

PV of \$1 million at the end of the first year = $\frac{\$1{,}000{,}000}{1.12} = \$892,857.14$

PV of \$1 million at the end of the second year = $\frac{\$1{,}000{,}000}{(1.12)^2} = \$797,193.88$

PV of \$1 million at the end of the third year = $\frac{\$1{,}000{,}000}{(1.12)^3} = \$711,780.25$

So the present value of the sum of these annual payment is $\$2,401,831.27$.

Exercise 12

Determine the present value of $25,000 to be received in the future in the following situations. In each case, assume the interest rate is 8%.

a) Payment is received at the end of one year's time given annual compounding.
b) Payment is received at the end of 20 years' time given annual compounding.
c) Payment is received at the end of one year's time given quarterly compounding (i.e., every three months).
d) Payment is received at the end of 20 years' time given quarterly compounding.
e) Payment is received at the end of one year's time given continuous compounding.
f) Payment is received at the end of 20 years' time given continuous compounding.

Solution:

Discrete compounding: $PV_t = \frac{V}{[1+(r/n)]^{nt}}$

Continous compounding: $PV_t = Ve^{-rt}$

Using these formulas we obtain:

(a) $$\frac{25,000}{1+0.08} = \frac{25,000}{1.08} = \$23,148.15$$

(b) $$\frac{25,000}{(1+0.08)^{20}} = \frac{25,000}{(1.08)^{20}} = \$5,363.71$$

(c) $$\frac{25,000}{[1+(0.08/4)]^{4}} = \frac{25,000}{(1.02)^{4}} = \$23,096.14$$

(d) $$\frac{25,000}{[1+(0.08/4)]^{80}} = \frac{25,000}{(1.02)^{80}} = \$5,127.74$$

(e) $$25,000e^{-0.08} = 25,000 \cdot 0.9231163 = \$23,077.91$$

(f) $$25,000e^{-1.6} = 25,000 \cdot 0.2018965 = \$5,047.41$$

Check your knowledge:

Test your knowledge with the following multiple-choice questions.

For each question, select the correct answer. Explain your decision.

Question 1

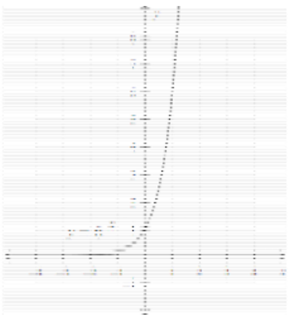
Determine the missing value in this table of values for the function $y = 2^x$.

x	$y = 2^x$
-1	0.5
0	
1	2

- **a) 1**
- b) -1
- c) 0
- d) 2

Question 2

Determine the range of $y = 6^x$.



- a) x > 0
- b) y is in the interval (-∞, ∞)
- c) y < 0
- **d) y > 0**

Question 3

Which exponential function is decreasing?

a) $\boldsymbol{y = \left(\frac{1}{3}\right)^x}$

b) $y = 7.7^x$

c) $y = 1.383^x$

d) $y = \left(\frac{5}{2}\right)^x$

Question 4

The expression log(x) represents the common logarithm of x. What is the value of the base of log(x)?

a) 1

b) 0

c) e

d) 10

Question 5

For which value of x is y = log(x) not defined??

a) X = -9

b) X = 1/9

c) X = 1

d) X = 81

Question 6

Which logarithm is equal to log(3x -1) – 5log(x)?

a) $\boldsymbol{log\left(\frac{3x-1}{x^5}\right)}$

b) $log\left(\frac{3x-1}{5x}\right)$

c) $log(x^5 + 3x - 1)$

Question 7

To increase a given present value, the discount rate should be adjusted:

a) upward.

b) downward.

Question 8

With continuous compounding at 10 percent for 30 years, the future value of an initial investment of $2,000 is closest to

a) $34,898.

b) $40,171.

c) $164,500.

d) $328,282.

Explanation: $FV = \$2000 * e^{(0.10*30)} = \$40.171$

Question 9

In 3 years, you are to receive $5,000. If the interest rate were to suddenly increase, the present value of that future amount to you would

a) fall.

b) rise.

c) remain unchanged.

d) cannot be determined without more information.

Question 10

What is the present value of £520,000 expected to be received in three years' time, if the business concerned requires a return of 10% on sums invested? Answers are given to the nearest £'000.

a) £692k

b) £432k

c) £473k

d) £390k

Explanation: $PV = \frac{FV}{(1+r)^t} = \frac{520000}{(1+0.1)^3} = 390683.7$

Question 11

Windsor Ltd is considering a project, which will involve the following cash inflows and (out)flows:

	£'000
Initial outlay	(400)
After one year	40
After two years	300
After three years	300

What will be the NPV (net present value) of this project if a discount rate of 15% is used?

a) +£58.8k

b) -£60.8k

c) +£240k

d) +460.8k

Explanation: $NPV = -C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} = -400 + \frac{40}{(1.15)^1} + \frac{300}{(1.15)^2} + \frac{300}{(1.15)^3}$=58.8