Queen Mary
University of London

Maths & Stats Pre-Sessional

Simple Functions and the basics of Present Value

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Simple Functions and the basics of Present Value

In this session:

- Definition of functions
- Commonly used functions and their applications in economics and finance:
  - Linear and quadratic functions
  - Exponential and logarithmic functions
- Definition of sequences and series
- Present value calculations

For more extensive reading, refer to Chapters 2 and 3 of Hoy, M., Livernois, J., McKenna, C., Rees, R., and Stengos, R. (2011). Mathematics for Economics, MIT Press, 3rd Edition

What is a function?

Definition. Given two sets X and Y, a **function** from X to Y is a rule that associates with each number of X, one and only one number of Y.

We can write the function as

$$y = f(x), \qquad x \in X.$$

y is often referred to as the *dependent variable* and x as the *independent variable*.

We also say y is the value of the function f at x.

- The **domain** of the function refers to the set of all possible values of x.
- The **range** of the function is the set of all possible values for $f(x)$.

Functions are generally defined by algebraic formulas.

What is a function?

Definition. Given two sets X and Y, a **function** from X to Y is a rule that associates with each number of X, one and only one number of Y.

Examples of functions again: $\mathrm{X} = \mathbb{R} \equiv \{the\ set\ of\ real\ numbers\}$

- Take any number in X and add 1 to it: $h(x) = x + 1 \in \mathbb{R}$
- Take any number in X and square it: $g(x) = x^2, x \in \mathbb{R}$
- Take any number in X and multiply it by 10: $f(x) = 10x , x \in \mathbb{R}$

Questions:

What is the domain of $f(x)$?
What is the range of $g(x)$?
What is the value of the function $h(x)$ at 2?

Simple Functions: Linear Functions

A **linear function** takes the form

$$y = ax + b, \qquad x \in \mathbb{R}.$$

The *parameter* a is called the **slope** and the *parameter* b is called the **intercept** term of the linear function.

Note the difference between a parameter and a variable.

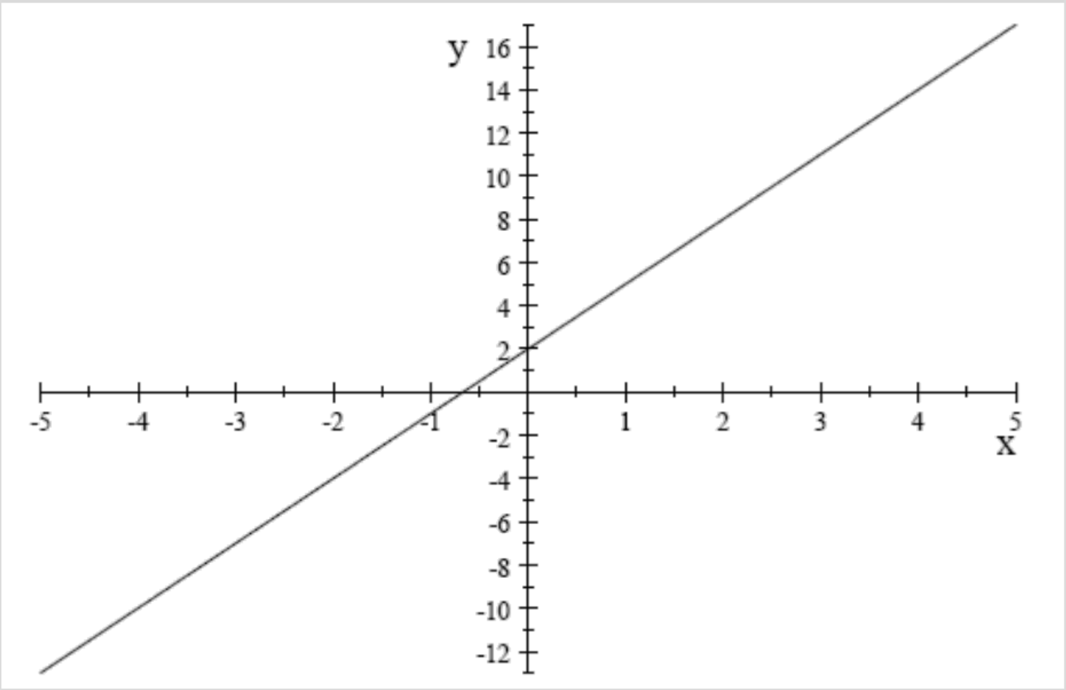
Example: consider $y = 2 + 3x$ and $y = 2 - 3x$,

what are the graphical representations of these two functions?

what are the slope and intercept of each of these functions?

Simple Functions: Linear Functions

<u>*Example*</u>: $y = 2 + 3x$

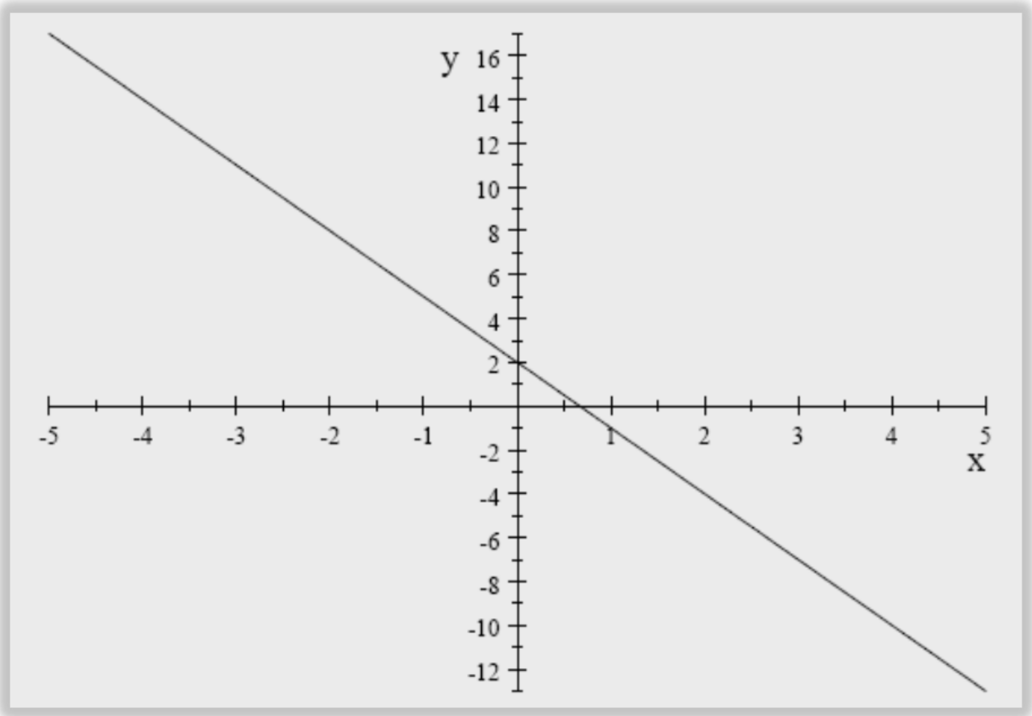


What is the graph of a function? How to draw $f(x) = 2 + 3x$, $x \in \mathbb{R}$?

- we want to draw a set of ordered pairs $\{(x, y)|\ y = 2 + 3x\}$ in the coordinate system x-y.
- Draw a few points, (-1,-1), (0,2), (2,8), in the x-y plane.
- Connect the points.

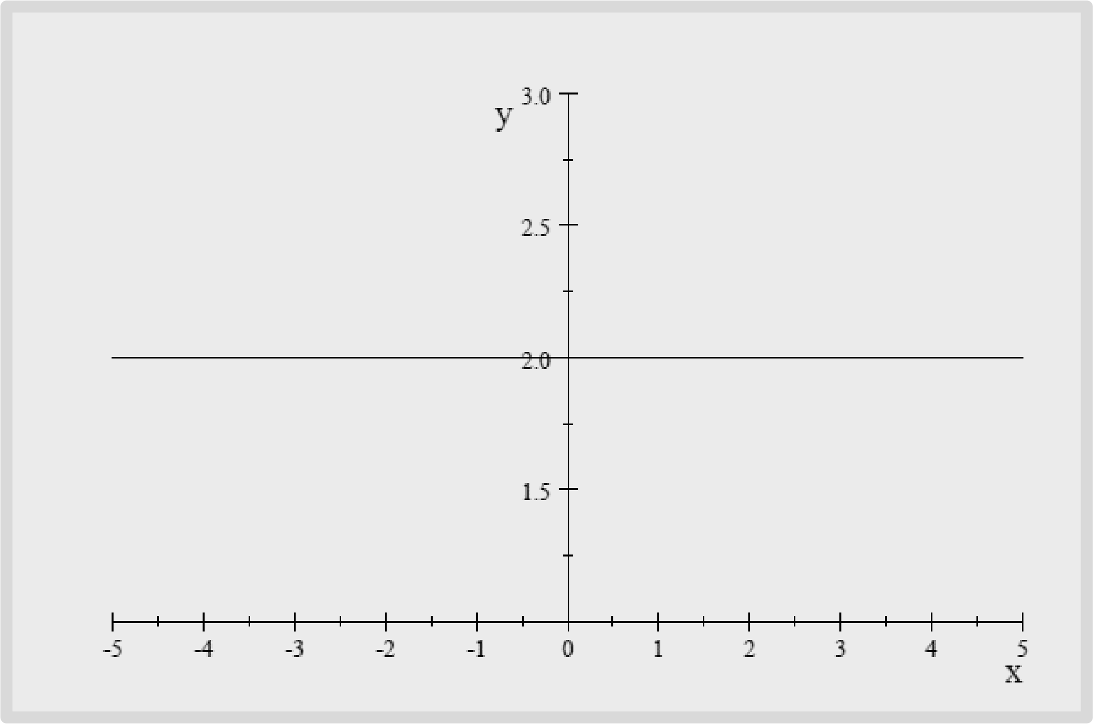
Simple Functions: Linear Functions

Example: $y = 2 - 3x$



$f(x)$

Example: $y = 2$



Simple Functions: Linear Functions

Given

$$y = ax + b, \quad x \in \mathbb{R}.$$

- For a > 0, an upward sloping line
- For a < 0, a downward sloping line
- For a= 0, a horizontal line

- For b > 0, the line intercepts the y-axis above the origin
- For b < 0, the line intercepts the y-axis below the origin
- For b = 0, the line goes through the origin.

Simple Functions: Linear Functions

- The graph of a linear function is a straight line.

- The slope of a line measures the change in y (Δy) divided by the change in x (Δx)

$$slope = \frac{\Delta y}{\Delta x}$$

The slope indicates the steepness and the direction of a line. The greater the absolute value, the steeper the line.

- The intercept is the point where the graph crosses the y axis and it occurs when $x = 0$.

Simple Functions: Quadratic Functions

A **quadratic function** takes the form

$$y = ax^2 + bx + c, \qquad x \in \mathbb{R}, \qquad a \neq 0.$$

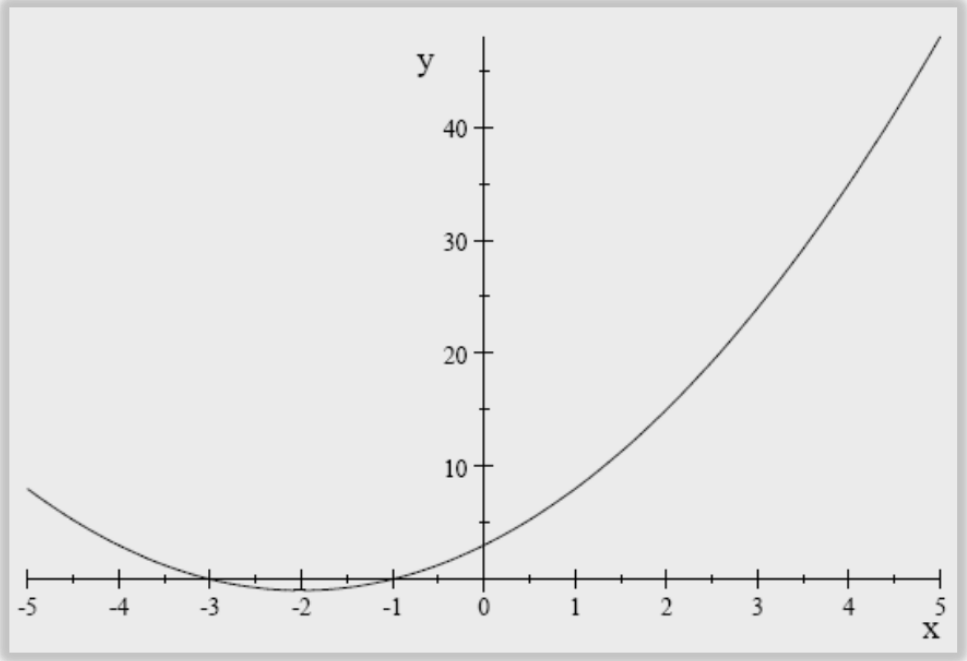
- If the parameter $a = 0$, the function becomes a linear function with slope b and intercept c.
- If the parameter $a > 0$, the graph of the function is U-shaped → *Convex function*
- If the parameter $a < 0$, the graph of the function is inverted U-shaped → *Concave function*

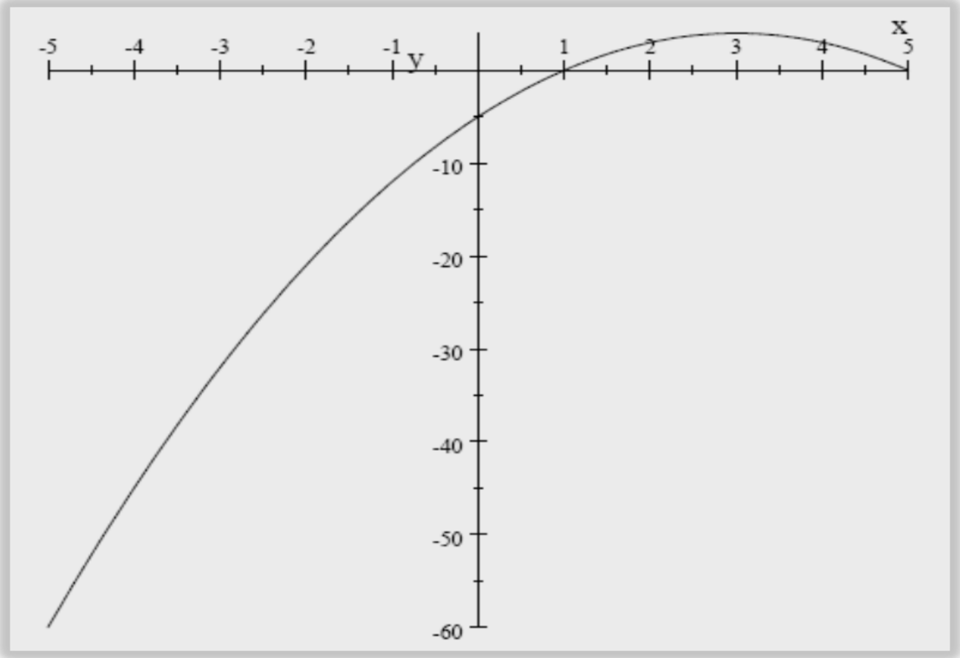
Quadratic Formula is the simplest way to find the roots of a quadratic equation:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Simple Functions: Quadratic Functions

<u>Example</u>: consider $y = x^2 + 4x + 3$ and $y = -x^2 + 6x - 5,$ which is U-shaped and which is inverted U-shaped?





Simple Functions: Quadratic Functions

Let the **quadratic function** be

$$y = ax^2 + bx + c, \qquad x \in \mathbb{R}, \qquad a \neq 0.$$

- Where does the graph intercepts with the x-axis?
- Equivalently, what are the roots of (or solutions to) the quadratic equation:

  $ax^2 + bx + c = 0$?

Answer: The Quadratic Formula:

$$\boxed{x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}}$$

Simple Functions: Exponential Functions

An **exponential function** takes the form

$$y = a^x, \qquad x \in \mathbb{R}.$$

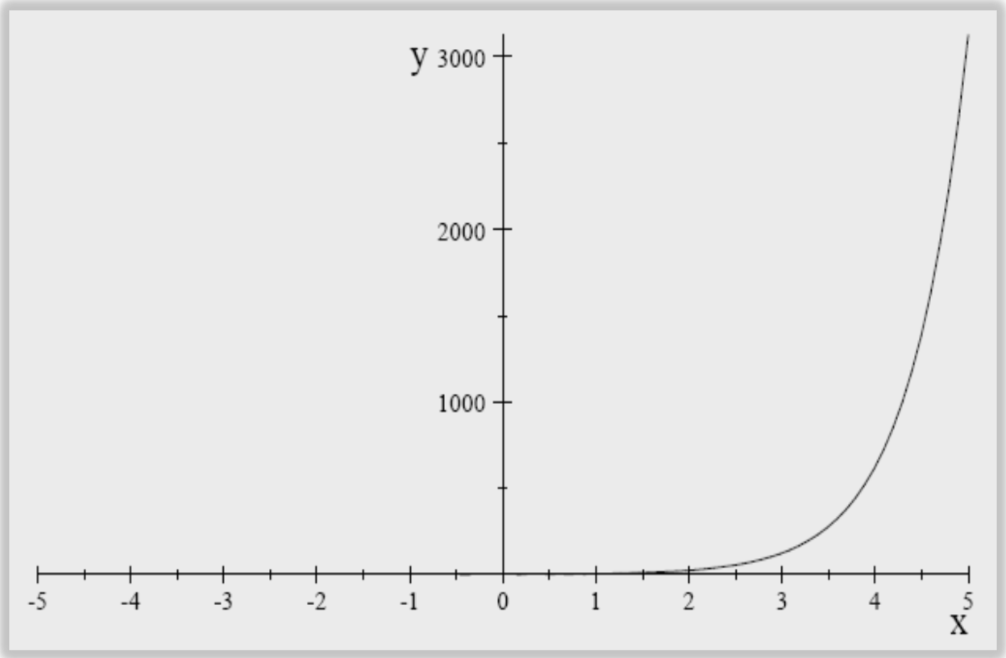
- The parameter a is called the base of the function and it is $a > 0$.
- If the parameter $a > 1$, the function increases in x and it is convex.
- If the parameter $0 < a < 1$, the function decreases in x and it is convex.

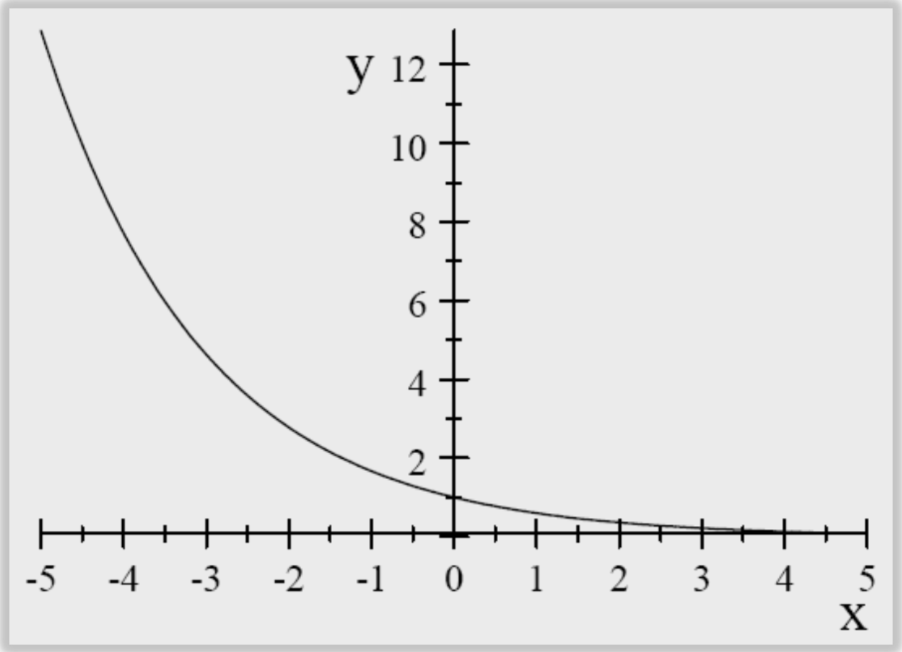
Examples $y = e^x$ where e = 2.71828. This function is called natural exponential function and it can also be written as $y = \exp(x)$.

Exponential functions are commonly used to express exponential growth such as interest compounding and depreciation.

Simple Functions: Exponential Functions

Example: consider $y = 5^x$ and $y = 0.6^x$





Simple Functions: Exponential Functions

Exponential functions are commonly used to express exponential growth such as interest compounding and depreciation. More on this later.

- You have just deposited £100 in your savings account this month. The monthly interest rate is 0.1%. Assuming no withdrawal, express the month-end balance of your account as a function of month t:

$$B(t) = 100 \times 1.001^t, \qquad t = 1,2,3, \ldots$$

- You have just bought a fax machine valued at £500. The value of the machine depreciates at 20% per year. Express the year-end value of the machine as a function of year t:

$$D(t) = 500 \times 0.8^t, \qquad t = 1,2,3, \ldots$$

Simple Functions: Logarithmic Functions

A logarithmic function takes the form:

$$y = \log_a x\,, \qquad x \in \mathbb{R}_+ \equiv \{a\ set\ of\ all\ positive\ numbers\}, a > 0, a \neq 1\ \ .$$

- If $a = e \approx 2.71828$, then we have the natural logarithm, write $y = \ln(x)$.
- If $a = 10$, then we have the common logarithm, write $y = \log(x)$.

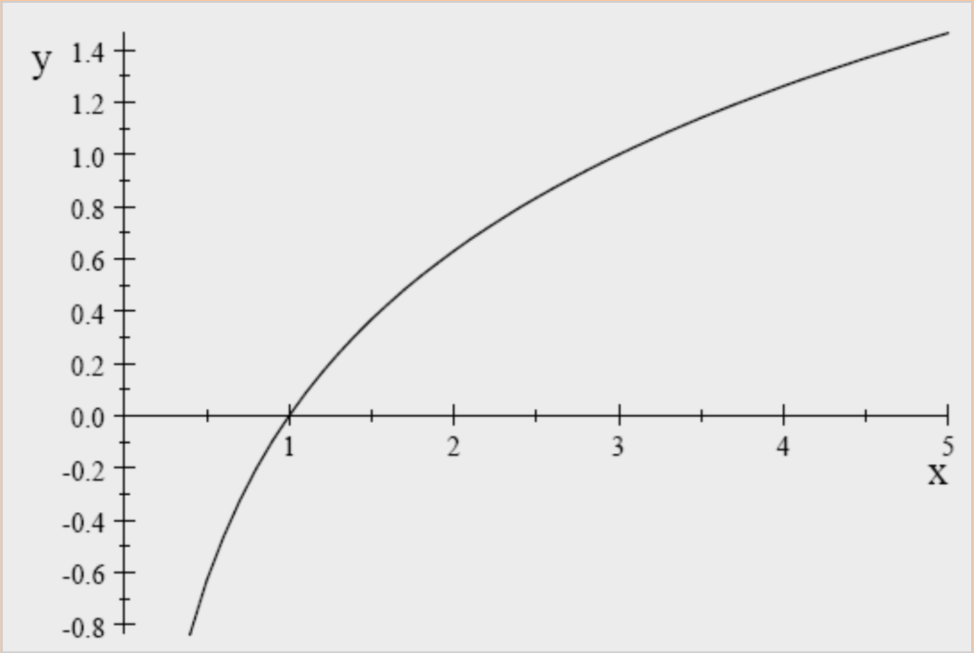
A log function is the inverse of the exponential function: $y = a^x \quad \Rightarrow x = \log_a x\,.$

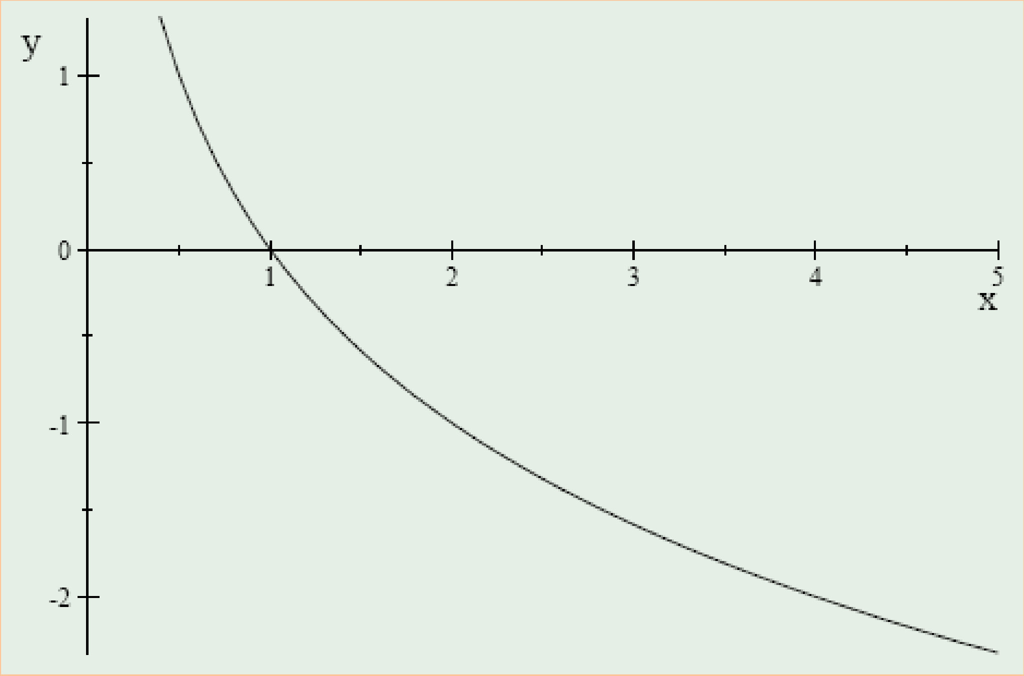
If the base $a > 1$, the function increases in x.

If the base $0 < a < 1$, the function decreases in x.

Simple Functions: Logarithmic Functions

Example: consider $y = \log_3 x$ and $y = \log_{0.5} x$.





Simple Functions: Logarithmic Functions

- Logarithmic Functions are commonly used to transform economic or financial variables.
- Take log of the Gross Domestic Product (GDP) to spot changes in growth rates.
- Take log of asset returns.

Example. Country A's GDP growth rate averages at 2% per year from 1950 to 1980 and accelerates to 10% per year from 1981 to 2010

Figure A: GDP Level (Normalized to 1 in 1950)

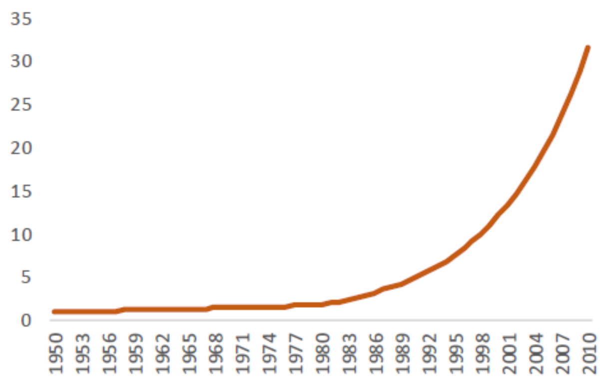
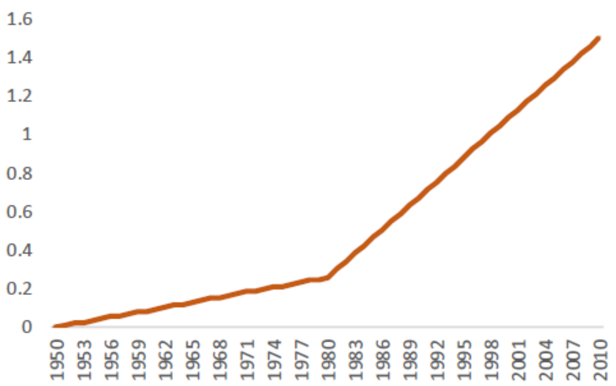


Figure B: log GDP (Normalized to 0 in 1950)



Simple Functions: Logarithmic Functions

- Properties of Logarithms

1. $log_a(xy) = log_a x + log_a y$
2. $log_a(\frac{x}{y}) = log_a x - log_a y$
3. $log_a x^n = n \log_a x$
4. $log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$
5. $log_a 1 = 0$
6. $log_a a = 1$
7. $log_a(a^m) = m$
8. $log_e(e^{bx}) = bx log_e e = bx * 1 = bx$

Simple Functions: Logarithmic Functions

Example

The stock price of company ABC varies over time as shown in the following table:

Time	1	2	3
Price	100	120	180

- The log return from time 1 to 2: log(120/100) = 7.92%
- The log return from time 2 to 3: log(180/120) = 17.62%
- The log return from time 1 to 3: log(180/100) = 25.53%
- Note: 7.92% + 17.62% = 25.53%! Time additive
- Simple returns don't have this property: 20% + 50% ≠ 80%.

Sequences

Definition:

- A **sequence** is a list of numbers (or elements) that exhibits a particular pattern or **function**.
- Each element in the sequence is called a **term**.
- A sequence can be **finite**, meaning it has a specific number of terms, or **infinite**, meaning it continues indefinitely.

Examples:

- $f(n) = 2n,\ n = 1, 2, 3, 4 \ldots$ or $2, 4, 6, 8, 10, \ldots$
- $f(n) = \frac{1}{n},\ n = 1, 2, 3, 4\ \ldots$ or $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

An important application of sequences in finance is the determination of the present value of a sum of money to be received at some time in the future

Future Value – Simple Interest

The **future value** of an asset refers to the amount of value that you estimate something will have at any point in the future.

Example: What will a machine be worth after 5 years? Or How much will my bank account will be worth in 6 months? You can measure both things using future value.

Future value using simple interest is calculated as $FV = PV(1 + rt)$

The equation shows that for any asset that earns fixed rate interest, the future value (**FV**) of the asset will be worth the present value (**PV**) multiplied by the function of interest rate (**r**) and time duration (**t**) plus 1.

Future Value – Simple Interest

Example:

You want to buy an investment for £100 that pays 1% interest each year and you plan on holding it hold it for 10 years. What is the future value of your investment?,

$$FV = £100(1 + 0.01 \times 10)$$

$$FV = £100(1.1)$$

$$FV = £110$$

Future Value – Simple Interest

Compound interest is similar to simple interest, except that accounts earning compound interest generate interest on the interest earned rather than just on the principal balance.

The future value of an investment that earns compound interest is calculated as:

$$FV = PV[(1+r)^t]$$

Example: You want to buy an investment for £100 that pays 10% interest each year and you plan on holding it hold it for 5 years. What is the future value of your investment?

$FV = £100[(1+10)^5]$

$FV = £100[1.61051] = £161.05$

Sequences and Present Value Calculations

Suppose you invest X pounds today at an annual rate of return r. Then at the end of the first year, your investment will generate

$$V = X (1 + r)$$

Then we say the present value of amount V to be received in one year's time is

$$PV = \frac{V}{1 + r} = X$$

Example. Suppose you will receive £120 at the end of one year and the annual interest rate is 20%. What is the present value of that £120?

$$PV = \frac{£120}{1 + 20\%} = £100$$

Sequences and Present Value Calculations

More generally, the present value PV_t of amount *V* to be received t periods from now when the interest rate is *r* per period and compounding occurs at the end of each period is

$$PV_t = \frac{V}{(1+r)^t}, \qquad t = 1, 2, 3, \ldots$$

- This is a sequence
- As t increases, PV_t decreases. The longer you wait, the more you discount future benefit
- We refer to $\frac{1}{1+r}$ as the discount factor

Sequences and Present Value Calculations

- What is a period? How frequently are interests compounded within a period?

Suppose a period is a year and the annual interest is 12%. You invest £1000 in the account for two years. Compute the value of your investment after two years in each of the following scenarios.

Scenario A: <u>Annual</u> compounding.

$$V = £1000(1 + 12\%)^2 = £1254.40$$

Scenario B: <u>Semiannual</u> compounding.

$$V = £1000(1 + 6\%)^4 = £1262.48$$

Scenario C: <u>Monthly</u> compounding.

$$V = £1000(1 + 1\%)^{24} = £1269.73$$

Sequences and Present Value Calculations

If we compound interest n times a year and the annual interest rate is r, then the value of £P invested for t years is

$$V = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we compound infinitely many times a year, or let $n \to +\infty$, we have the case of **continuous compounding**, and the value of £P invested at interest rate r for t periods is worth

$$V = Pe^{rt}$$

at the end of t periods.

Scenario D: <u>Continuous</u> compounding.

$$V = £1000e^{12\% \times 2} = £1271.25$$

Sequences and Present Value Calculations

The present value of £V to be received t years from now is given by the following formulae:

- If the interest rate is r per year and compounding n times per year

$$PV_t = \frac{V}{(1 + {}^{r}/_{n})^{nt}}$$

- If the interest rate is r per year under continuous compounding

$$PV_t = Ve^{-rt}.$$

Series

Definition:

- A **series** is the cumulative sum of a given sequence of terms.
- A **series** can be highly generalized as the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms of the sequence.

Example:

$$PV_T = \sum_{t=1}^{T} \frac{V}{(1+r)^t} = \frac{V}{1+r} + \frac{V}{(1+r)^2} + \cdots + \frac{V}{(1+r)^T}$$

Series and NPV

Example:

Dexable Inc. is planning a project with an initial investment of $45,000 and expects to generate $30,000 annually for two years, with an 8% discount rate. Is it worth investing in the project?

$$NPV = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} - C_0$$

$$NPV = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} - C_o$$

Where C_0=initial investment; C = cash flow; r = discount rate and t = time

NPV = ($27,777.78 + $25,720.17) - $45,000 = $8,497.95

Series and NPV

How do we decide whether a project is worth taking up or not?

- If the NPV of a project is negative, then the project is not worthwhile
- If the NPV of a project is positive, then the project is worth investing in