Queen Mary
University of London

Main Examination period 2024 – January – Semester A

MTH6151 / MTH6151P: Partial Differential Equations Solutions

Comments on questions:

- 1(a) is a test of concept-understanding based on the lectures.
- 1(b) is a variant of lecture notes examples.
- 1(c) is similar to problem set questions.
- 1(d) is a verification of basic definitions of heat equations.
- 2 is similar to problem set questions.
- 3(a) and 3(b) are both variants of lecture notes examples.
- 4(a) is a test of basic concepts.
- 4(b) is a test of understanding of the general solutions of Laplace equations, may require some good understanding of the solutions' structure.
- 4(c) is a test of students' ability to apply the right properties of harmonic functions
- 5(a) is a test of computation of derivatives and verifying a function satisfying heat equations.
- 5(b) is similar to lecture notes examples.

Question 1 [26 marks].

(a) For each of the following equations, write down the order of the equation, determine whether each of them is linear or non-linear, and say whether they are homogeneous or inhomogeneous.

(i) $e^x \Delta U - y^{2024} U_{xxy} = 0.$

(ii) $(1 + U_x^2)U - U_x U_y = 0.$ **[6]**

Solution:

(i) This is a 3rd order, linear, homogeneous equation.

(ii)This is a 1st order, non-linear, homogeneous equation.

(b) Find the general solutions $U(x, t)$ for the PDE

$$U_{xx} - 5U_{xt} + 4U_{tt} = 0.$$

[10]

Solution:

First, we can factor the equation as

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t})U = 0.$$

Denote by $W = (\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t})U$, we then get 2 PDEs:

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})W = 0 \quad (1)$$
$$(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t})U = W \quad (2)$$

Solve (1) as a first order linear PDE using either characteristics or change of coordinates, we get

$$W(x, t) = f(x + t).$$

The characteristic lines for equation (2) is $t = -4x + C$, and along the characteristic lines, the PDE becomes an ODE

$$\frac{d}{dx}U(x, t(x)) = U_x - 4U_t = f(x + t).$$

Integrate both sides, we get

$$U(x, t) = F(x + t) + G(C).$$

Using the characteristic equation $C = 4x + t$, we get the general solutions

$$U(x, t) = F(x + t) + G(4x + t),$$

for any F, G.

(c) Consider the following 2nd order PDE,

$$U_{xx} + 2U_{xy} + 4U_x = 0.$$

(i) Write down its principal part and then determine the type (elliptic, parabolic or hyperbolic).

(ii) Change the principal part of the above equation to a canonical form (i.e. without cross-derivatives).

[**6**]

Solution:

(i) The principal part is $U_{xx} + 2U_{xy}$. $1^2 - 0 > 0$ and thus the equation is hyperbolic.

(ii) Using the change of variables $x' = x, y' = -x + y$, we get the principal part becomes $U_{x'x'} - U_{y'y'}$.

(d) Find all possible values of a, b, c so that $U(x,t) = ax^2 + bt + ct^2$ solves the heat equation

$$U_t - \varkappa U_{xx} = 0, \varkappa > 0.$$

[**4**]

Solution:

Plug into the equation, we get $(b + 2ct) - \varkappa \cdot 2a = 0$ for any t.

Thus we must have $c = 0$ and $b = 2\varkappa a$.

Question 2 [20 marks].

(a) Consider the following eigenvalue problem:

$$X'' + \lambda X = 0$$
$$X'(0) = 0, X(\pi) = 0.$$

(i) Show that all the eigenvalues satisfy $\lambda > 0$.

(ii) Find all eigenvalues and eigenfunctions. [**8**]

Solution:

Multiply both sides by X and do integration by parts, we get

$$\int_0^\pi [X \cdot X'' + \lambda X^2] = 0$$
$$X \cdot X'|_0^\pi - \int_0^\pi (X')^2 + \lambda \int_0^\pi X^2 = 0.$$

The boundary conditions imply $X \cdot X'|_0^\pi = 0$, and thus $\lambda > 0$.

The solution to the eigenvalue problem is $X(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. It's derivative is $X'(x) = -c_1\sqrt{\lambda} \sin \sqrt{\lambda}x + c_2\sqrt{\lambda} \cos \sqrt{\lambda}x$.

The first boundary condition implies $0 = X'(0) = c_2\sqrt{\lambda}$, so $c_2 = 0$.

The second boundary condition then implies $c_1 \cos \sqrt{\lambda}\pi = 0$, which forces $\sqrt{\lambda}\pi = \frac{\pi}{2} + n\pi$, for $n = 1, 2 \ldots$.

We get the eigenvalues $\lambda_n = (\frac{1}{2} + n)^2$ and the eigenfunctions $X_n = \cos[(\frac{1}{2} + nx)]$.

(b) Solve the following wave equation with mixed boundary conditions on an interval. (You can make use of the results obtained in (a).)

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U_x(0,t) = 0, U(\pi, t) = 0 \\ U(x,0) = 0, U_t(x,0) = 6c \cdot \cos(\frac{3}{2}x). \end{cases}$$

[**12**]

Solution:

First we consider solutions of the form $U(x,t) = X(x)T(t)$ and get the equation

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda.$$

The initial conditions give the eigenvalue problem in (a)

$$X'' + \lambda X = 0$$
$$X'(0) = 0, X(\pi) = 0.$$

We get the eigenvalues $\lambda_n = (\frac{1}{2} + n)^2$ and the eigenfunctions $X_n = \cos[(\frac{1}{2} + nx)]$.

Knowing λ_n, we can solve for T using $\ddot{\ } T + \lambda c^2 T = 0$ and get

$$T_n = a_n \cos[(\frac{1}{2} + n)ct] + b_n \sin[(\frac{1}{2} + n)ct].$$

the general solutions are

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} a_n \cos[(\frac{1}{2} + nx)] \cos[(\frac{1}{2} + n)ct] + \sum_{n=1}^{\infty} b_n \cos[(\frac{1}{2} + nx)] \sin[(\frac{1}{2} + n)ct]. \end{aligned}$$

Differentiate it we get

$$U_t(x,t) = \sum_{n=1}^{\infty} -a_n \cdot \cos[(\frac{1}{2} + nx)] \sin[(\frac{1}{2} + n)ct] + \sum_{n=1}^{\infty} b_n \cdot (\frac{1}{2} + n)c \cos[(\frac{1}{2} + nx)] \cos[(\frac{1}{2} + n)ct].$$

The first initial value tells us $a_n = 0$ for any n.
The second initial value tells us $6c \cdot \cos \frac{3x}{2} = \sum_{n=1}^{\infty} b_n \cdot (\frac{1}{2} + n)c \cos[(\frac{1}{2} + nx)]$.
We then see $b_n = 0$ for any $n \neq 1$ and $b_1 = \frac{6c}{\frac{3}{2}c} = c$.
So the solution is

$$U(x,t) = 4 \cos \frac{3x}{2} \sin \frac{3ct}{2}.$$

Question 3 [16 marks].

(a) Solve the inhomogeneous 1st order equation

$$\begin{aligned} &U_x - U_t = \cos t \\ &U(x,0) = 0. \end{aligned}$$

[**8**]

Solution:

The characteristic lines are $t = -x + C$. Along the characteristic lines, the PDE become and ODE

$$\frac{d}{dx}U(x,t(x)) = \cos t.$$

Using the characteristic equation, we get $t = -x + C$ and thus $\frac{dU}{dx} = \cos(-x + C)$. Integrate it we get $U(x,t) = -\sin(-x + C) + f(C)$. Plug back in the characteristic equations $C = t + x$ we get the solution

$$U(x,t) = -\sin t + f(t + x).$$

Next, using $U(x,0) = 0$, we get $-\sin 0 + f(0 + x) = 0$, namely $f(x) \equiv 0$. So the solution to this PDE is

$$U(x,t) = -\sin t.$$

(b) Solve the inhomogeneous wave equation on the real line

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = \sin x, x \in \mathbb{R} \\ &U(x,0) = 0, U_t(x,0) = 0. \end{aligned}$$

Explain what theory you are using and show your full computations. [**8**]

Solution:

Using the Duhamel's principle for inhomogeneous wave equations, we get

$$\begin{aligned} U(x,t) =& \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} \sin r dr ds \\ =& \frac{1}{2c}\int_0^t [-\cos(x+ct-cs) + \cos(x-ct+cs)]ds \\ =& \frac{1}{2c}[-\sin(x+ct-cs)]\frac{1}{-c}|_0^t + \frac{1}{2c}[\sin(x-ct+cs)]\frac{1}{c}|_0^t \\ =& \frac{1}{2c^2}\sin x - \frac{1}{2c^2}\sin(x+ct) + \frac{1}{2c^2}\sin x - \frac{1}{2c^2}\sin(x-ct) \\ =& \frac{1}{c^2}\sin x - \frac{1}{2c^2}[\sin(x+ct) + \sin(x-ct)] \end{aligned}$$

Question 4 [20 marks].

(a) What form does the Laplace equation take in polar coordinates (r, θ)? **[2]**

Solution:

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

(b) Let (r, θ) denote the usual polar coordinates. Show that if $U(r, \theta)$ is a harmonic function, then so is $V(r, \theta) = U(\frac{1}{r}, -\theta)$. **[6]**

Solution:

If U is harmonic, then it must be of the form of the general solution in polar coordinates

$$U(r, \theta) = (C_0 + D_0 \ln r) + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m} \right) (A_m \cos m\theta + B_m \sin m\theta).$$

And thus

$$\begin{aligned} V(r, \theta) &= U(\frac{1}{r}, -\theta) \\ &= \big(C_0 + D_0 \ln(r^{-1})\big) + \sum_{m=1}^{\infty} \left(C_m r^{-m} + \frac{D_m}{r^{-m}} \right) (A_m \cos(-m\theta) + B_m \sin(-m\theta)) \\ &= \left(C_0 + \hat{D}_0 \ln r\right) + \sum_{m=1}^{\infty} \left(\hat{C}_m r^m + \frac{\hat{D}_m}{r^m} \right) (\hat{A}_m \cos m\theta + \hat{B}_m \sin m\theta), \end{aligned}$$

where $\hat{D}_0 = -D_0$, and $\hat{C}_m = D_m, \hat{D}_m = C_m, \hat{A}_m = A_m, \hat{B}_m = -B_m$ for any $m = 1, 2 \ldots$.

So $U(\frac{1}{r}, -\theta)$ also satisfies the form of general solutions to Laplace equations in polar coordinates and thus is harmonic.

(c) Suppose that U is a solution to the Laplace equation in the disk $\Omega = \{r \leq 1\}$ and that $U(1, \theta) = 5 - \sin^2 \theta$.

(i) Without finding the solution to the equation, compute the value of U at the origin – i.e. at $r = 0$.

(ii) Without finding the solution to the equation, determine the location of the maxima and minima of U in Ω.

(Hint: $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$.) **[12]**

Solution:

(i) By the mean value theorem, the value at the origin is equal to the average on the boundary

$$U(0) = \frac{1}{2\pi}\int_0^{2\pi} 5 - \sin^2 \theta d\theta = \frac{1}{2\pi}\int_0^{2\pi} (5 - \frac{1}{2}) d\theta + \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta = \frac{9}{2}.$$

(ii) By the maximum principle, the maximum and minimum value in the disk all happen at the boundary. And at the boundary, we see
$U(1,\theta) = 5 - \sin^2\theta = \frac{9}{2} + \frac{\cos 2\theta}{2}$.

The maximum happens at the polar coordinates $(1, 0)$ and $(1, \pi)$ with values $\frac{9}{2} + \frac{1}{2} = 5$.

The minimum happens at the polar coordinates $(1, \frac{\pi}{2})$ and $(1, \frac{3\pi}{2})$ with values $\frac{9}{2} - \frac{1}{2} = 4$.

Question 5 [18 marks].

(a) Show that $V(x,t) = \pi - \int_0^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2} ds$ is a solution to the heat equation

$$V_t = \varkappa V_{xx}, x \in \mathbb{R}.$$

[6]

Solution:

We compute the derivatives:

$$\begin{aligned}
V_x &= -e^{-\frac{x^2}{4\varkappa t}} \cdot \frac{1}{\sqrt{4\varkappa t}} \\
V_{xx} &= e^{-\frac{x^2}{4\varkappa t}} \cdot \frac{x}{2\varkappa t \cdot \sqrt{4\varkappa t}} \\
V_t &= -e^{-\frac{x^2}{4\varkappa t}} \cdot \frac{-1}{2} \frac{x}{\sqrt{4\varkappa} \cdot t^{\frac{3}{2}}}.
\end{aligned}$$

Thus $V_t = \varkappa V_{xx}$ and the heat equation is satisfied.

(b) Suppose U solves the heat equation on the real line

$$U_t = 4U_{xx}, x \in \mathbb{R}$$

with initial value

$$U(x,0) = f(x) = \begin{cases} 4, x \leq 0 \\ 2, x > 0. \end{cases}$$

(i) Use the Fourier-Poisson formula to give an explicit expression for the solution U.

(ii) Describe the qualitative behaviour of U in this case as $t \to \infty$ and plot out the solution at several instants of time to explain your answer. What is the limit of U as $t \to \infty$?

[12]

Solution:

(i) Using the Fourier-Poisson formula, we get

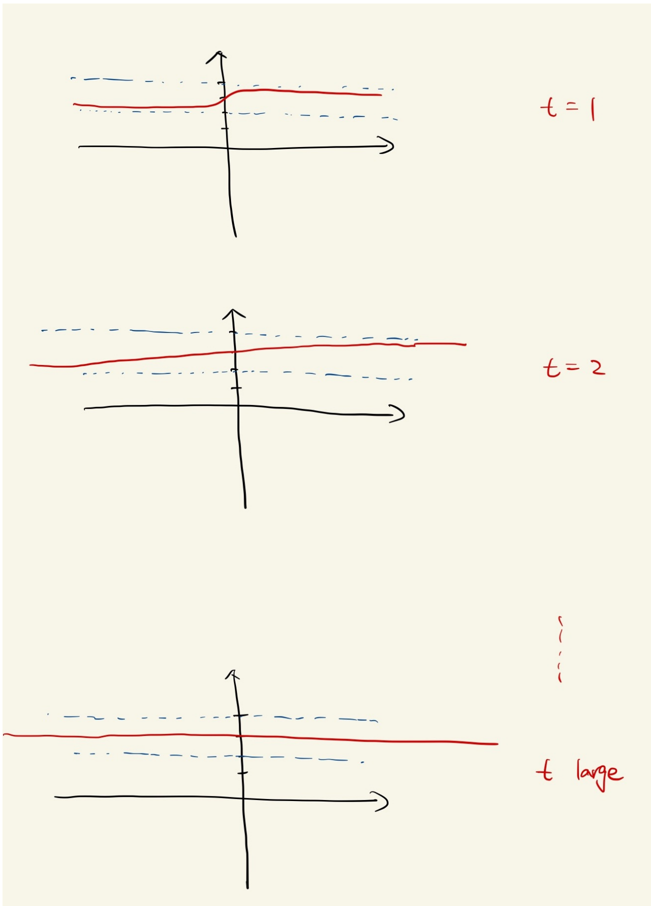
$$\begin{aligned}
U(x,t) &= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y) dy \\
&= 2\int_0^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} dy + 4\int_{-\infty}^{0} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} dy.
\end{aligned}$$

By a change of variables $s = \frac{x-y}{\sqrt{4\varkappa t}} \Rightarrow dy = -\sqrt{4\varkappa t}ds$, we get

$$\begin{aligned}
U(x,t) &= -2\int_{\frac{x}{\sqrt{4\varkappa t}}}^{-\infty} e^{-s^2}ds - 4\int_{\infty}^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2}ds \\
&= 2\int_{-\infty}^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2}ds + 4\int_{\frac{x}{\sqrt{4\varkappa t}}}^{\infty} e^{-s^2}ds \\
&= 2\int_{-\infty}^{\infty} e^{-s^2}ds + 2\int_{\frac{x}{\sqrt{4\varkappa t}}}^{\infty} e^{-s^2}ds \\
&= 2\int_{-\infty}^{\infty} e^{-s^2}ds + 2\int_{0}^{\infty} e^{-s^2}ds - 2\int_{0}^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2}ds \\
&= 3 - 2\int_{0}^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2}ds
\end{aligned}$$

And as $t \to \infty$, we see $U(x,t) \to 3$.

The plot is as follows



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