Queen Mary
University of London

Main Examination period 2024 – January – Semester A

MTH6151 / MTH6151P: Partial Differential Equations

Examiners: Shengwen Wang, Oliver Jenkinson

You will have a period of **3 hours** to complete the exam. You have additional **30 minutes** for scanning and submitting your solution.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [26 marks].

(a) For each of the following equations, write down the order of the equation, determine whether each of them is linear or non-linear, and say whether they are homogeneous or inhomogeneous.

(i) $e^x \Delta U - y^{2024} U_{xxy} = 0$.

(ii) $(1 + U_x^2)U - U_x U_y = 0$. [**6**]

(b) Find the general solutions $U(x, t)$ for the PDE

$$U_{xx} - 5U_{xt} + 4U_{tt} = 0.$$

[**10**]

(c) Consider the following 2nd order PDE,

$$U_{xx} + 2U_{xy} + 4U_x = 0.$$

(i) Write down its principal part and then determine the type (elliptic, parabolic or hyperbolic).

(ii) Change the principal part of the above equation to a canonical form (i.e. without cross-derivatives).

[**6**]

(d) Find all possible values of a, b, c so that $U(x, t) = ax^2 + bt + ct^2$ solves the heat equation

$$U_t - \varkappa U_{xx} = 0, \varkappa > 0.$$

[**4**]

Question 2 [20 marks].

(a) Consider the following eigenvalue problem:

$$X'' + \lambda X = 0$$
$$X'(0) = 0, X(\pi) = 0.$$

(i) Show that all the eigenvalues satisfy $\lambda > 0$.

(ii) Find all eigenvalues and eigenfunctions. [**8**]

(b) Solve the following wave equation with mixed boundary conditions on an interval. (You can make use of the results obtained in (a).)

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0 \\ U_x(0,t) = 0, U(\pi,t) = 0 \\ U(x,0) = 0, U_t(x,0) = 6c \cdot \cos(\frac{3}{2}x). \end{cases}$$

[**12**]

Question 3 [16 marks].

(a) Solve the inhomogeneous 1st order equation

$$U_x - U_t = \cos t$$
$$U(x, 0) = 0.$$

[**8**]

(b) Solve the inhomogeneous wave equation on the real line

$$U_{tt} - c^2 U_{xx} = \sin x, x \in \mathbb{R}$$
$$U(x, 0) = 0, U_t(x, 0) = 0.$$

Explain what theory you are using and show your full computations. [**8**]

Question 4 [20 marks].

(a) What form does the Laplace equation take in polar coordinates (r, θ)? [**2**]

(b) Let (r, θ) denote the usual polar coordinates. Show that if $U(r, \theta)$ is a harmonic function, then so is $V(r, \theta) = U(\frac{1}{r}, -\theta)$. [**6**]

(c) Suppose that U is a solution to the Laplace equation in the disk $\Omega = \{r \leq 1\}$ and that $U(1, \theta) = 5 - \sin^2 \theta$.

(i) Without finding the solution to the equation, compute the value of U at the origin – i.e. at $r = 0$.

(ii) Without finding the solution to the equation, determine the location of the maxima and minima of U in Ω.

(Hint: $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$.) [**12**]

Question 5 [18 marks].

(a) Show that $V(x,t) = \pi - \int_0^{\frac{x}{\sqrt{4\varkappa t}}} e^{-s^2} ds$ is a solution to the heat equation

$$V_t = \varkappa V_{xx}, x \in \mathbb{R}.$$

[**6**]

(b) Suppose U solves the heat equation on the real line

$$U_t = 4U_{xx}, x \in \mathbb{R}$$

with initial value

$$U(x,0) = \begin{cases} 4, x \leq 0 \\ 2, x > 0. \end{cases}$$

(i) Use the Fourier-Poisson formula to give an explicit expression for the solution U.

(ii) Describe the qualitative behaviour of U in this case as $t \to \infty$ and plot out the solution at several instants of time to explain your answer. What is the limit of U as $t \to \infty$?

[**12**]

End of Paper.