PROBLEM SET 9 FOR MTH 6151

1. Show that if $U(x,t)$ is a solution to the heat equation then $U(\alpha x, \alpha^2 t)$, α a constant, is also a solution to the heat equation. What about $U(\alpha x, -\alpha^2 t)$, is this a solution too?

2. Find all the values of the constants a and b such that

$$U(x,t) = e^{ax+bt}$$

satisfies the heat equation.

3. Use the Fourier-Poisson formula to compute the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = 1. \end{aligned}$$

Interpret the result you obtain. Is this surprising?

4. Use the Fourier-Poisson formula to find the limit as $t \to \infty$ of the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = \begin{cases} 1 & -L < x < L \\ 0 & x < -L, \quad x > L \end{cases}. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

5. Use the Fourier-Poisson formula to find the limit as $t \to \infty$ of the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases}. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

6. Use the Fourier-Poisson formula to compute the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = e^{3x}. \end{aligned}$$

7. Use the Fourier-Poisson formula to compute the solution to the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = \begin{cases} 0 & x < 0 \\ e^{-x} & x > 0 \end{cases}. \end{aligned}$$

What happens as $t \to \infty$.