WEEK 10 NOTES

1. The heat equation on the real line

In this section we will see how to solve the problem

$$U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0,$$
$$U(x, 0) = f(x).$$

That is, we want to solve the heat equation on the real line given that we know the initial form of U.

In order to solve this problem we will need some further assumptions on the solution $U(x, t)$ and the initial data $f(x)$. In particular, we want $U(x, t)$ to be absolutely integrable —that is,

$$\int_{-\infty}^{\infty} |U(x,t)| dx < \infty. \tag{1.1}$$

Also, we require that U and U_x go to zero at infinity —that is,

$$U(x,t),\, U_x(x,t) \longrightarrow 0, \qquad x \to \pm\infty.$$

Note. *Observe that to have condition* (1.1) *one needs* $U(x,t)$ *to go to zero at infinity.*

We also require f to be absolutely integrable:

$$\int_{-\infty}^{\infty} f(x) dx < \infty.$$

Example 1.1. Functions which are absolutely integrable are special —i.e. not all functions are absolutely integrable. Some examples are:

(i) $f(x) = \sin x$. One then has that

$$\int_{-\infty}^{\infty} |\sin x| dx = \infty.$$

That is, $\sin x$ is not integrable.

(ii) Let

$$f(x) = \frac{1}{1+x^2}.$$

One has that

$$\int_{-\infty}^{\infty} \left| \frac{1}{1+x^2} \right| dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi < \infty.$$

Thus $f(x) = 1/(1+x^2)$ is absolutely integrable.

(iii) Let $f(x) = e^{-x^2}$. From Calculus we know that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty,$$

so, again, absolutely integrable.

For solutions to the heat equations with finite total heat, we have the following property.

Lemma 1.2. *Let $U(x,t)$ be a solution to the heat equation which is absolutely integrable and satisfying $U_x(x,t) \to 0$ as $x \to \pm\infty$. Then*

$$\int_{-\infty}^{\infty} U(x,t)dx$$

is constant for $t \geq 0$.

Proof. To see this integrate the heat equation over the real line:

$$\int_{-\infty}^{\infty} U_t(x,t)dx = \varkappa \int_{-\infty}^{\infty} U_{xx}(x,t)dx.$$

This can be rewritten, using the Fundamental Theorem of Calculus as

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} U(x,t)dx &= \varkappa U_x(x,t)\Big|_{-\infty}^{\infty} \\ &= 0.\end{aligned}$$

The last equality follows from the requirement that U_x goes to zero at infinity. Thus, the integral

$$\int_{-\infty}^{\infty} U(x,t)dx$$

does not depend on t —that is, it is constant. □

Let's first consider the following very special solutions of heat equations. (Indeed, it will help us generate all the other solutions).

Theorem 1.3. *The function*

$$U(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}$$

solves the heat equation $U_t = \varkappa U_{xx}$ on the real line.

Proof. We compute

$$\begin{aligned}\partial_t U =& \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi}} \cdot \frac{-1}{2t^{\frac{3}{2}}} + \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} \cdot \frac{x^2}{4\kappa t^2} \\ =& \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} \cdot [-\frac{1}{2t} + \frac{x^2}{4\kappa t^2}].\end{aligned}$$

And

$$\begin{aligned}\partial_{xx} U =& \partial_x[\frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi}} \cdot \frac{-x}{2\varkappa t}] \\ =& \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi}} \cdot \frac{x^2}{4\kappa^2 t^2} + \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi}} \cdot \frac{-1}{2\kappa t} \\ =& \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi}}[-\frac{x}{2\varkappa t} + \frac{x^2}{4\kappa^2 t^2}].\end{aligned}$$

Thus

$$U_t = \varkappa U_{xx}.$$

□

This solution is known as the *heat kernel* or *fundamental solution* of the heat equation. We denote it by:

$$K(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}.$$

We note the following properties:

(i) By construction $K(x,t)$ satisfies the heat equation. That is,

$$K_t = \varkappa K_{xx}, \qquad x \in \mathrm{R}, \quad t > 0.$$

(ii) The heat kernel is an even function —that is, $K(x,t) = K(-x,t)$.

(iii) $K(x,t)$ is a smooth function (i.e. C^∞) for $x \in \mathbb{R}$, $t > 0$.

(iv) One has that

$$\int_{-\infty}^{\infty} K(x,t)dx = 1, \qquad t \geq 0.$$

(v) For $x \neq 0$ one has that

$$K(x,t) \to 0, \qquad \text{as} \quad t \to 0^+,$$

while for $x = 0$ one has that

$$K(0,t) \to \infty, \qquad \text{as} \quad t \to 0^+.$$

(vi) For any $x \in \mathbb{R}$, one has

$$K(x,t) \to 0, \qquad \text{as} \quad t \to \infty$$

Remark 1.4. Properties *(i)*-*(iv)* above, follow from the construction of the heat Kernel given in the previous section. Only property *(v)* requires further work. If $x \neq 0$ then to compute the limit it is enough to consider

$$\frac{e^{-1/t}}{\sqrt{t}} = \frac{1}{\sqrt{t}e^{1/t}} \longrightarrow 0 \qquad \text{as} \quad t \to 0^+,$$

given that $e^{1/t} \to \infty$ and recalling that the exponential grows faster than any power of t so it dominates $\sqrt{t}$. For $x = 0$ one has that

$$K(0,t) = \frac{e^0}{\sqrt{4\pi\varkappa t}} = \frac{1}{\sqrt{4\pi\varkappa t}} \longrightarrow \infty \qquad \text{as} \quad t \to 0^+.$$

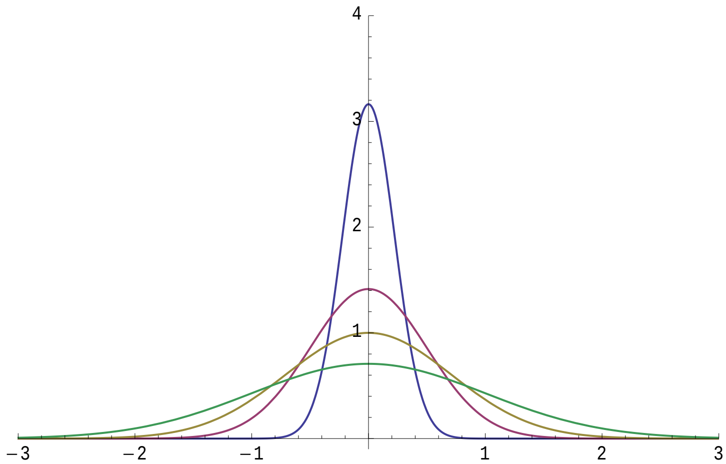
Note. *Property* (v) *together with* (iv) *show that $K(x,0)$ is a very special object —in fact, it turns out that $K(x,0)$ cannot be a function. It is a more general type of object known as* generalised function *or* distribution*.*

2. Dirac's delta function

In order to better understand the properties of the heat Kernel at $t = 0$ consider the sequence of functions

$$\{f_\lambda(x)\} = \left\{ \frac{e^{-\frac{x^2}{\lambda^2}}}{\lambda\sqrt{\pi}} \right\}, \qquad \lambda \in \mathbb{R}^+.$$

Graphs of the functions f_λ for various values of λ can be seen in the figure below:



Observe that as $\lambda \to 0$, the Gaussian bells become increasingly peaked. One can then check that:

(i) if $x \neq 0$ then $f_\lambda(x) \to 0$ as $\lambda \to 0$;
(ii) if $x = 0$ then $f_\lambda(0) \to \infty$ as $\lambda \to 0$;
(iii) moreover, one has that

$$\int_{-\infty}^{\infty} f_\lambda(x)dx = 1$$

for all λ so that, in particular, one has

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} f_\lambda(x)dx = 1.$$

The limit of the family $\{f_\lambda\}$ is not a proper function. However, one can *formally* write

$$\delta(x) \equiv \lim_{\lambda \to 0} f_\lambda(x) = \lim_{\lambda \to 0} \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}}.$$

This is the so-called *Dirac's delta "function"*.

Note. *There is a branch of mathematics known as* distribution theory *aimed at making sense of objects like Dirac's delta.*

Definition 2.1. Dirac's delta, δ, is defined by the conditions:

(i) $\delta(x) = 0$ for $x \neq 0$;
(ii) $\delta(0) = \infty$;
(iii) for any $a < 0 < b$ one has

$$\int_a^b \delta(x) = 1.$$

Note. *From the previous discussion it follows that*

$$K(x, 0) = \delta(x).$$

In terms of diffusion processes, $\delta(x)$ describes an infinitesimally small "drop" of ink concentrated at the origin. This "drop" then spreads with time.

3. The general solution to the heat equation on the real line

The heat kernel is the basic building block to obtain the general solution to the heat equation on the real line.

We begin by observing the following property:

Lemma 3.1. *If $U(x,t)$ is a solution to*

$$U_t = \varkappa U_{xx}$$

then

$$V(x,t) \equiv \int_{-\infty}^{\infty} U(x-y,t)g(y)dy \tag{3.1}$$

is also a solution for any function g —as long as the integral converges.

Proof. This follows by direct computation:

$$V_t(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_t(x-y,t)g(y)dy,$$

$$V_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_{xx}(x-y,t)g(y)dy.$$

Hence,

$$V_t(x,t) - \varkappa V_{xx}(x,t) = \int_{-\infty}^{\infty}\Big(U_t(x-y,t) - \varkappa U_{xx}(x-y,t)\Big)g(y)dy = 0.$$

□

Note. *The operation given by* (3.1) *is called the* convolution *of U and g. This is sometimes denoted as*

$$V(x,t) = (U * g)(x,t).$$

Now, consider the problem

$$U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{3.2}$$

$$U(x,0) = f(x). \tag{3.3}$$

Claim: the (unique) solution to (3.2)-(3.3) is given by

$$U(x,t) = \int_{-\infty}^{\infty} K(x-y,t)f(y)dy,$$

with K denoting the heat kernel. Or, more explicitly,

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y)dy. \tag{3.4}$$

The latter is known as the *Fourier-Poisson formula*.

As a consequence of Lemma 3.1, and given that $K(x,t)$ satisfies the heat equation, then $U(x,t)$ as defined by (3.4) is a solution to the heat equation.

Note. *To fully address the claim it is only necessary to verify that $U(x,0) = f(x)$.*

3.0.1. *Some auxiliary calculations.* In the following it will be convenient to consider the function

$$Q(x,t) \equiv \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varkappa t}} e^{-s^2} ds, \qquad t > 0. \tag{3.5}$$

Observe that

$$\begin{aligned} Q_x(x,t) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left(\frac{x}{\sqrt{4\varkappa t}} \right) e^{-\frac{x^2}{4\varkappa t}} \\ &= \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\pi\varkappa t}} = K(x,t). \end{aligned}$$

Thus, $Q(x,t)$ is the antiderivative (with respect to x) of $K(x,t)$.

Next, we consider the limit of $Q(x,t)$ as $t \to 0^+$. There are 2 cases:

(i) $x > 0$. Here we have

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1.$$

(ii) $x < 0$. Here one has

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds = \frac{1}{2} - \frac{1}{2} = 0.$$

Hence, one concludes that

$$\lim_{t\to 0^+} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \equiv H(x).$$

The function H defined above is called *Heaviside's step function*.

Note. *As Q is the antiderivative of K it follows from the above discussion that*

$$H'(x) = \delta(x).$$

That is, Dirac's delta is the derivative of Heaviside's step function.

3.0.2. *Concluding the main computation.* Using the properties of Q as discussed in the previous subsection one has that

$$\begin{aligned} U(x,t) &= \int_{-\infty}^{\infty} K(x-y,t) f(y) dy \\ &= \int_{-\infty}^{\infty} Q_x(x-y,t) f(y) dy \\ &= -\int_{-\infty}^{\infty} Q_y(x-y,t) f(y) dy \\ &= -Q(x-y,t) f(y) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t) f'(y) dy, \end{aligned}$$

where in the third line one makes use of the chain rule to change the x-derivative to a y-derivative and in the fourth line one employs integration by parts to pass the derivative from Q to f. Now, as $K(x-y,t)$ decays very fast to 0 as $|x-y| \to \infty$ it follows that

$$-Q(x-y,t) f(y) \Big|_{-\infty}^{\infty} = 0.$$

Hence,

$$U(x,t) = \int_{-\infty}^{\infty} Q(x-y,t) f'(y) dy.$$

We make use of this expression to compute the limit $t \to 0^+$:

$$\begin{aligned} U(x,0^+) &= \int_{-\infty}^{\infty} Q(x-y,0^+) f'(y) dy = \int_{-\infty}^{\infty} H(x-y) f'(y) dy \\ &= \int_{-\infty}^{x} f'(y) dy = f(y)\Big|_{-\infty}^{x} = f(x), \end{aligned}$$

where in the last line it has been assumed that $f(x) \to 0$ as $x \to -\infty$.

We summarise the previous discussion in the following:

Proposition 3.2. *For $t > 0$, the* ***Fourier-Poisson formula***

$$\boxed{U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y) dy}$$

gives the (unique) solution to

$$\begin{cases} U_t(x,t) = \varkappa U_{xx}(x,t), & x \in \mathbb{R}, \quad t > 0, \\ U(x,0) = f(x). \end{cases}$$

3.1. **Some examples.** In this section we discuss some examples of computation involving the Fourier-Poisson formula.

Example 3.3. Analyse the behaviour of the solution $U(x,t)$ given by the Fourier-Poisson formula in the case

$$f(x) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

In this case one has

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4\varkappa t} H(y) dy \\ &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{0}^{\infty} e^{-(x-y)^2/4\varkappa t} dy \end{aligned}$$

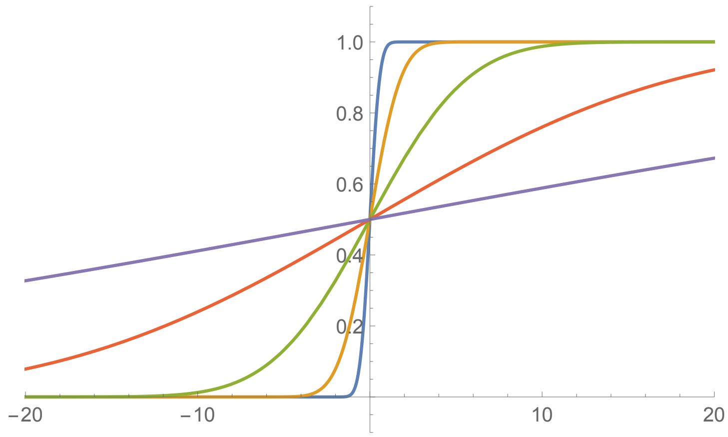
as $H(x) \neq 0$ only for $x > 0$. Letting now

$$s = \frac{x-y}{\sqrt{4\varkappa t}} \Longrightarrow dy = -\sqrt{4\varkappa t} ds,$$

one finds that

$$\begin{aligned} U(x,t) &= -\frac{\sqrt{4\varkappa t}}{\sqrt{4\pi\varkappa t}} \int_{x/\sqrt{4\varkappa t}}^{-\infty} e^{-s^2} ds \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\varkappa t}} e^{-s^2} ds \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds + \frac{1}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2} ds \\ &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2} ds. \end{aligned}$$

Thus, observe that, in fact
$$U(x,t)=Q(x,t).$$
We now investigate the behaviour of $U(x,t)$ for fixed x as $t\to\infty$:
$$\lim_{t\to\infty}U(x,t)=\frac{1}{2}+\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4\varkappa t}}e^{-s^2}ds=\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^0 e^{-s^2}ds=\frac{1}{2},$$
as $x/\sqrt{4\varkappa t}\to 0$ as $t\to\infty$.

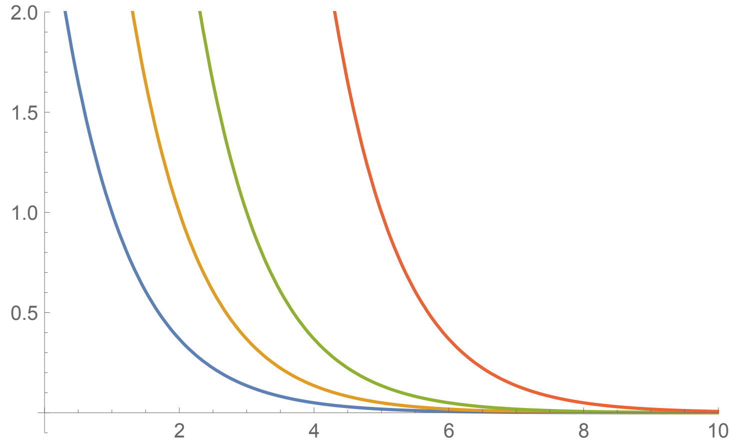


Example 3.4. Evaluate the Fourier-Poisson formula in the case
$$f(x)=e^{-x}.$$
Substituting the above expression in the formula one obtains
$$U(x,t)=\frac{1}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-(x-y)^2/4\varkappa t}e^{-y}dy.$$
The exponent in the integral can be rearranged as
$$\begin{aligned}-\frac{(x-y)^2}{4\varkappa t}-y&=-\frac{x^2-2xy+y^2+4\varkappa ty}{4\varkappa t}\\&=-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}+\varkappa t-x.\end{aligned}$$
Hence,
$$\begin{aligned}U(x,t)&=\frac{1}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}+\varkappa t-x}dy\\&=\frac{e^{\varkappa t-x}}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}}dy.\end{aligned}$$
Letting
$$s=\frac{y+2\varkappa t-x}{\sqrt{4\varkappa t}}\Longrightarrow ds=\frac{dy}{\sqrt{4\varkappa t}},$$
it follows then that
$$\begin{aligned}U(x,t)&=\frac{e^{\varkappa t-x}}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty}e^{-s^2}\sqrt{4\varkappa t}ds\\&=\frac{e^{\varkappa t-x}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-s^2}ds=e^{\varkappa t-x}.\end{aligned}$$

Observe, in particular, that

$$U(x,t) \to \infty, \qquad \text{as} \quad t \to \infty.$$

Thus, the solution does not decay but grows at every point x. Plots of this solution for various values of t are shown below.



We conclude the list of examples with one particular solution to the heat equation which evidences an important property of the heat equation:

Example 3.5. Compute the solution to the heat equation on the real line if the initial condition is given by

$$f(x) = \frac{e^{x^2/4\varkappa}}{\sqrt{4\pi\varkappa}}.$$

In this case substitution of the initial condition into the Fourier-Poisson formula gives

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} \frac{e^{\frac{y^2}{4\varkappa}}}{\sqrt{4\pi\varkappa}} dy \\ &= \frac{1}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{\frac{y^2}{4\varkappa} - \frac{(x-y)^2}{4\varkappa t}} dy. \end{aligned}$$

The exponent in the integral can be manipulated by completing squares to get

$$\begin{aligned} \frac{y^2}{4\varkappa} - \frac{(x-y)^2}{4\varkappa t} &= \frac{1}{4\varkappa}\left(y^2 - \frac{x^2}{t} - \frac{y^2}{t} + \frac{2xy}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(1 - \frac{1}{t}\right)y^2 + \frac{2xy}{t} - \frac{x^2}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y^2 + \frac{2xy}{t-1} + \frac{x^2}{(t-1)^2}\right) - \frac{x^2}{t} - \frac{x^2}{t(t-1)}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y + \frac{x}{t-1}\right)^2 - \frac{x^2}{t-1}\right). \end{aligned}$$

Hence,

$$U(x,t) = \frac{1}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{\frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y+\frac{x}{t-1}\right)^2 - \frac{x^2}{t-1}\right)} dy$$
$$= \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\varkappa}\left(\frac{1-t}{t}\right)\left(y+\frac{x}{t-1}\right)^2} dy.$$

Finally, letting

$$s = \sqrt{\frac{1-t}{4\varkappa t}}\left(y + \frac{x}{t-1}\right) \Longrightarrow ds = \sqrt{\frac{1-t}{4\varkappa t}} dy,$$

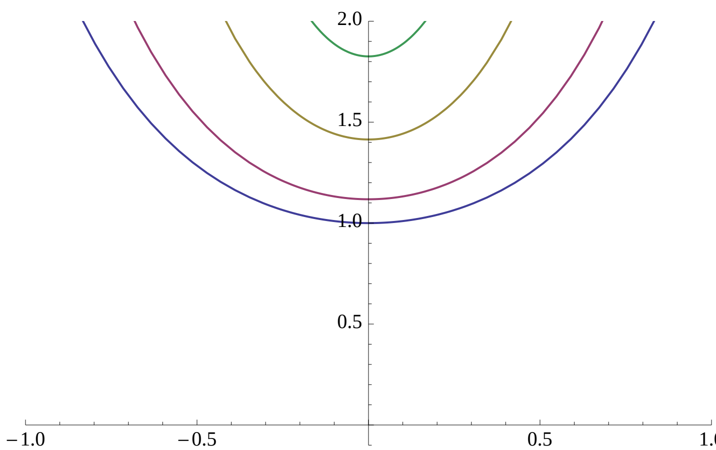
one concludes that

$$U(x,t) = \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}} \sqrt{\frac{4\varkappa t}{1-t}} \int_{-\infty}^{\infty} e^{-s^2} ds$$
$$= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\pi\sqrt{4\varkappa(1-t)}} \sqrt{\pi}$$
$$= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\sqrt{4\varkappa\pi(1-t)}}.$$

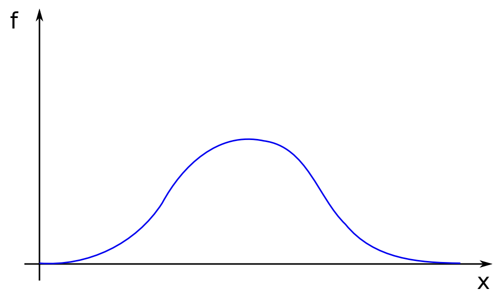
Observe that

$$U(x,t) \longrightarrow \infty \qquad \text{as} \quad t \to 1.$$

That is, the solution becomes singular in a finite amount of time! A plot of the solutions for various values of t is shown below:



Example 3.6. In this example we suppose the initial date f is a bump function so that $f = 0$ for $|x| \geq R_0$ and $|f(x)| \leq C_0$. See for example the of a bump function graphed below.



Then by the Fourier-Poisson formula, we get

$$\begin{aligned}U(x,t) =&\frac{1}{\sqrt{4\pi\varkappa t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} f(y)dy\\ =&\frac{1}{\sqrt{4\pi\varkappa t}}\int_{-R_0}^{R_0} e^{-\frac{(x-y)^2}{4\varkappa t}} f(y)dy\\ \leq&\frac{1}{\sqrt{4\pi\varkappa t}}\cdot 2R_0\cdot C_0\\ \to&0, \text{ as } t\to\infty.\end{aligned}$$

Here in the second line we used that $f = 0$ for $x \geq R_0$ or $x \leq -R_0$.

We see that after a very long time, the solution to the heat equation tend to zero!

3.2. **Invariance properties of the heat equation.** An important property of the heat equation involves the behaviour of its solutions with respect to scalings of the coordinates. More precisely,

Lemma 3.7. *If $U(x,t)$ solves the heat equation then also $V(x,t) \equiv U(ax, a^2t)$ also solves the heat equation.*

Proof. Let $v = ax$, $w = a^2t$. Then, using the chain rule one finds that

$$\begin{aligned}U_t(v,w) &= \frac{\partial w}{\partial t}U_w(v,w) = a^2U_w(v,w),\\ U_x(v,w) &= \frac{\partial v}{\partial x}U_v(v,w) = aU_v(v,w),\\ U_{xx}(v,w) &= a^2U_{vv}(v,w).\end{aligned}$$

Hence,

$$U_t(ax,a^2t) - \varkappa U_{xx}(ax,a^2t) = a^2\Big(U_w(v,w) - \varkappa U_{vv}(v,w)\Big) = 0.$$

□