Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6107 Chaos & Fractals

Duration: 2 hours

Date and time: 30th April 2014, 14.30–16.30

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): O. Jenkinson

Question 1 (a) Suppose we are given a non-empty set Σ and a map $f : \Sigma \to \Sigma$.

(i) [1 mark] What does it mean to say that $x \in \Sigma$ is a *fixed point* for f?

(ii) [2 marks] What does it mean to say that $x \in \Sigma$ is a *periodic point* for f?

(iii) [1 mark] How is the *minimal period* of a periodic point defined?

(iv) [2 marks] What does it mean to say that $x \in \Sigma$ is an *eventually periodic point* for f?

(v) [6 marks] Prove that if f is invertible then every eventually periodic point is a periodic point.

(b) [5 marks] Give a detailed statement of Sharkovsky's Theorem.

(c) Suppose the map $f : [0, 1] \to [0, 1]$ is defined by

$$f(x) = \begin{cases} x + 1/2 & \text{for } x \in [0, 1/2) \\ 2 - 2x & \text{for } x \in [1/2, 1]\,. \end{cases}$$

(i) [3 marks] For this map f, determine all its fixed points.

(ii) [4 marks] For this map f, determine an eventually periodic point which is not periodic.

(iii) [4 marks] For this map f, determine all its points of minimal period 2.

Question 2 (a) [2 marks] For a differentiable map $f : \mathbb{R} \to \mathbb{R}$, how is the *multiplier* of a periodic orbit defined?

(b) [2 marks] Write down a condition on the multiplier which guarantees that a periodic orbit is *stable* (i.e. *attractive*).

(c) Let $f_\lambda : [-1, 1] \to [-1, 1]$ be the logistic map, defined by $f_\lambda(x) = 1 - \lambda x^2$ for parameters $\lambda \in [0, 2]$.

(i) [3 marks] For $\lambda \in [0, 2)$, compute the fixed point $x^* = x^*(\lambda) \in [-1, 1]$ of f_λ.

(ii) [3 marks] Compute the multiplier of this fixed point $x^*(\lambda)$.

(iii) [2 marks] Determine the largest value λ_1 with the property that the fixed point $x^*(\lambda)$ is stable for all $\lambda \in [0, \lambda_1)$.

(iv) [6 marks] For $\lambda > \lambda_1$, determine the periodic orbit of f_λ which has minimal period 2.

(v) [4 marks] Compute the multiplier of this period-2 orbit, and determine the largest value λ_2 with the property that this orbit is stable for all $\lambda \in (\lambda_1, \lambda_2)$.

(vi) [2 marks] Briefly define what is meant by a *period-doubling bifurcation*.

(vii) [3 marks] How is the *Feigenbaum constant* δ defined?

Question 3 (a) [6 marks] Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be

(i) a *homeomorphism*,

(ii) a *diffeomorphism*,

(iii) *order preserving.*

(b) [7 marks] Prove that an order preserving diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ does not have any points of minimal period strictly larger than 1.

Question 4 (a) [4 marks] Let $C_0 = [0, 1]$. In the standard construction of the Cantor ternary set $C = \cap_{k=0}^{\infty} C_k$, describe briefly how the sets C_k are defined.

(b) [2 marks] Write down the sets C_1 and C_2.

(c) [2 marks] If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(d) [2 marks] What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(e) [4 marks] Given a set $A \subset \mathbb{R}$, how is its *box dimension* defined?

(f) [4 marks] Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}$. For $A, B \in \mathcal{H}$, how is the *Hausdorff distance* $h(A, B)$ defined?

(g) [4 marks] Compute $h(C_1, C_2)$.

(h) [4 marks] Using your answers to parts (c) and (d), or otherwise, show that if the box dimension of the ternary Cantor set $C \subset \mathbb{R}$ exists then it must equal $\log 2/ \log 3$.

(i) [3 marks] Given two maps $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$, how is the corresponding *iterated function system* $\Phi : \mathcal{H} \to \mathcal{H}$ defined?

(j) [3 marks] Write down two maps $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ such that the ternary Cantor set C is the fixed point of the corresponding iterated function system Φ.

End of Paper