Queen Mary
University of London

MTH6107 / MTH6107P: Chaos & Fractals

Duration: 2 hours

Date and time: 19th May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): O. Jenkinson

Question 1. [27 marks]

(a) For a differentiable map $f : \mathbb{R} \to \mathbb{R}$, how is the **multiplier** of a periodic orbit defined? **[2]**

(b) Write down a condition on the multiplier which guarantees that a periodic orbit is **stable** (i.e. **attractive**). **[2]**

(c) Let $f_\lambda : [-1, 1] \to [-1, 1]$ be the logistic map, defined by $f_\lambda(x) = 1 - \lambda x^2$ for parameters $\lambda \in [0, 2]$.

(i) For $\lambda \in [0, 2)$, compute the fixed point $x^* = x^*(\lambda) \in [-1, 1]$ of f_λ. **[3]**

(ii) Compute the multiplier of this fixed point $x^*(\lambda)$. **[3]**

(iii) Determine the largest value λ_1 with the property that the fixed point $x^*(\lambda)$ is stable for all $\lambda \in [0, \lambda_1)$. **[2]**

(iv) For $\lambda > \lambda_1$, determine the periodic orbit of f_λ which has minimal period 2. **[6]**

(v) Compute the multiplier of this period-2 orbit, and determine the largest value λ_2 with the property that this orbit is stable for all $\lambda \in (\lambda_1, \lambda_2)$. **[4]**

(vi) Briefly define what is meant by a **period-doubling bifurcation**. **[2]**

(vii) How is the **Feigenbaum constant** δ defined? **[3]**

Question 2. [26 marks]

(a) Given a subset of $\mathbb{R}^2$, how is its **box dimension** defined? **[4]**

(b) Briefly describe the construction of the **Sierpinski triangle** P^*. Use this description to show that if the box dimension of P^* exists then it must equal $\log 3/ \log 2$. **[8]**

(c) Let $\mathcal{H}$ denote the collection of compact subsets of $\mathbb{R}^2$. For $A, B \in \mathcal{H}$, how is the **Hausdorff distance** $h(A, B)$ defined? **[4]**

(d) Given a finite collection of self-maps of $\mathbb{R}^2$, how is the corresponding **iterated function system** defined? **[4]**

(e) What does it mean for a self-map of $\mathbb{R}^2$ to be a **contraction mapping**? **[3]**

(f) State the Dubins & Freedman Theorem on iterated function systems consisting of contraction mappings. **[3]**

Question 3. [25 marks]
Let Σ denote the interval $[-1, 1]$.

(a) Explain what it means for two maps $f, g : \Sigma \to \Sigma$ to be **topologically conjugate**. **[3]**

(b) Show that the notion of topological conjugacy defines an equivalence relation on the set of self-maps of Σ. **[4]**

(c) Use the map $h(x) = \sin(\pi x/2)$ to show that the map $f : \Sigma \to \Sigma$ defined by $f(x) = 1 - 2|x|$ is topologically conjugate to the Ulam map $g : \Sigma \to \Sigma$ given by $g(x) = 1 - 2x^2$. **[6]**

(d) Find the fixed point of the map $G : \Sigma \to \Sigma$ defined by $G(x) = 1 - x^2$, and determine, with justification, whether this point is unstable, stable, or superstable. **[4]**

(e) Find the periodic orbit of minimal period 2 for G, and determine, with justification, whether this orbit is unstable, stable, or superstable. **[4]**

(f) Determine whether the map $F : \Sigma \to \Sigma$ given by $F(x) = 1 - |x|$ is topologically conjugate to G, being careful to justify your answer. **[4]**

Question 4. [22 marks]
Let $\sigma : [0, 1) \to [0, 1)$ and $\tau : [0, 1) \to [0, 1)$ be defined by $\sigma(x) = 2x \pmod 1$ and $\tau(x) = 3x \pmod 1$.

(a) Given $x \in [0, 1)$, with binary expansion $x = \sum_{k=1}^{\infty} b_k/2^k$ where each $b_k \in \{0, 1\}$, show that x is periodic under σ if and only if the binary digit sequence $(b_k)_{k=1}^{\infty}$ is periodic. **[10]**

(b) Determine the period-5 orbit of σ which is contained in the interval $[3/20, 13/20]$. **[3]**

(c) Determine the periodic orbit of σ which is contained in the interval $[3/10, 4/5]$. **[3]**

(d) Identify, with justification, those points of minimal period 4 for σ which are also of minimal period 4 for τ. **[6]**

End of Paper.