Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH6107 / MTH6107P: Chaos & Fractals

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: O. Jenkinson, S. Wang

Throughout this exam, the notation $d_1d_2d_3d_4d_5d_6d_7d_8d_9$ will denote your 9-digit QMUL student ID number, where d_i is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for $1 \le i \le 9$. The distinct digits in your student ID number, written in increasing order, will be denoted $a_1 < a_2 < a_3 < \ldots < a_k$, where $k \le 9$ is the cardinality of the set $\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9\}$.
Your largest student digit will be denoted $a = a_k = \max\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9\}$.

Question 1 [25 marks]. Given an iterated function system defined by the two maps $\phi_1(x) = (x + a_1)/10$ and $\phi_2(x) = (x + a_4)/10$, define $\Phi(A) = \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0, 1])$ for $k \ge 0$.

(a) Write down a_1 and a_4, and determine the sets C_1 and C_2. **[4]**

(b) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k. **[3]**

(c) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k? **[3]**

(d) Compute the box dimension of $C = \cap_{k=0}^{\infty} C_k$, being careful to justify your answer. **[5]**

(e) Compute the box dimension of $D = \cap_{k=0}^{\infty} \Psi^k([0, 1])$, where $\Psi(A) = \psi_1(A) \cup \psi_2(A)$, and $\psi_1(x) = (x + a_1)/16$, $\psi_2(x) = (x + a_4)/16$. **[5]**

(f) Describe a set E whose box dimension is equal to $a_4/(a_4 + 1)$, being careful to justify your answer. **[5]**

Question 2 [25 marks]. Let $a = \max\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9\}$ be your largest student ID digit, and suppose the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - a$.

(a) Determine all fixed points of f, and determine whether each fixed point is attracting or repelling, taking care to justify your answer. **[5]**

(b) Determine all 2-cycles for f, and determine whether each 2-cycle is attracting or repelling, taking care to justify your answer. **[5]**

(c) Give one example of an eventually fixed point that is not itself a fixed point, and one example of an eventually periodic point of least period 2 that is not itself a periodic point. **[5]**

(d) If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 + a$, determine whether there is a topological conjugacy from f to g, taking care to justify your answer. **[5]**

(e) If $F : \mathbb{R} \to \mathbb{R}$ and and $G : \mathbb{R} \to \mathbb{R}$ are defined by $F(x) = x - a$ and $G(x) = x + a$, determine whether there is a topological conjugacy from F to G, taking care to justify your answer. **[5]**

Question 3 [25 marks]. For parameters $\lambda > 0$, define $f_\lambda : \mathbb{R} \to \mathbb{R}$ by

$$f_\lambda(x) = \lambda x^2(1-x)\,.$$

(a) Show that there is a point $p \in \mathbb{R}$ which is a fixed point of f_λ for all $\lambda > 0$. Is p attracting or repelling? Justify your answer. **[5]**

(b) Determine the value $\lambda_1 > 0$ such that f_λ has precisely one fixed point if $\lambda \in (0, \lambda_1)$, and precisely 3 fixed points if $\lambda > \lambda_1$. Justify your answer. **[5]**

(c) For $\lambda > \lambda_1$, let $x_\lambda^- < x_\lambda^+$ denote the two fixed points of f_λ which are not equal to p. Determine explicit formulae for x_λ^- and x_λ^+ in terms of λ. **[2]**

(d) Show that x_λ^- is a repelling fixed point of f_λ for all $\lambda > \lambda_1$. **[5]**

(e) Determine the value $\lambda_2 > \lambda_1$ such that if $\lambda \in (\lambda_1, \lambda_2)$ then x_λ^+ is an attracting fixed point of f_λ, and if $\lambda > \lambda_2$ then x_λ^+ is a repelling fixed point of f_λ. Justify your answer. **[5]**

(f) Show that there exists $\lambda \in (5, 6)$ such that $2/3$ is a point of least period 2 for f_λ. **[3]**

Question 4 [25 marks].

(a) For the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x + \sum_{i=1}^{9} x^{2d_i+1}$, give a formula for the derivative $f_1'(x)$. **[2]**

Using properties of this derivative, or otherwise, show that the only periodic point for f_1 is the fixed point at 0. **[8]**

(b) For the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x - 2 & \text{for } x \geq 0\,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_2 \text{ has a point of least period } n\}$, being careful to justify your answer. **[5]**

(c) For the function $f_3 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_3(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x/2 - 2 & \text{for } x \geq 0\,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_3 \text{ has a point of least period } n\}$, being careful to justify your answer. **[5]**

(d) Without using Sharkovskii's Theorem, show that every continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic orbit must have a fixed point. [*Hint: Use the Intermediate Value Theorem.*] **[5]**

End of Paper.