Queen Mary University of London

Main Examination period 2024 – January – Semester A

MTH6107: Chaos & Fractals

Examiners: O. Jenkinson, R. Klages

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper (i.e., 6 faces in total)** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

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Question 1 [24 marks].

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 3x^2 + 3x$.

(i) Determine all fixed points of f. **[3]**

(ii) Determine, with justification, whether each fixed point is attracting or repelling. **[3]**

(iii) Determine the basin of attraction of each attracting fixed point. **[3]**

(iv) Give an example of an eventually periodic orbit that is not periodic, or explain why such points do not exist. **[3]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$.

(i) Determine, with justification, whether f is topologically conjugate to $g_1 : \mathbb{R} \to \mathbb{R}$ defined by $g_1(x) = x + 2$. **[3]**

(ii) Determine, with justification, whether f is topologically conjugate to $g_2 : \mathbb{R} \to \mathbb{R}$ defined by $g_2(x) = x - 1$. **[3]**

(iii) Determine, with justification, whether f is topologically conjugate to $g_3 : \mathbb{R} \to \mathbb{R}$ defined by $g_3(x) = -x + 1$. **[3]**

(iv) Determine, with justification, whether f is topologically conjugate to $g_4 : \mathbb{R} \to \mathbb{R}$ defined by $g_4(x) = x^2 - 1$. **[3]**

Question 2 [27 marks]. For parameters $\lambda \in [0, 1]$, define $f_\lambda : [0, 1] \to [0, 1]$ by $f_\lambda(x) = \lambda \sin(\pi x)$.

(a) Sketch the graphs of the functions $f_{1/4}$ and f_1. **[2]**

(b) Determine the value $\lambda_1 \in (0, 1)$ such that the fixed point 0 is attracting for $\lambda \in [0, \lambda_1)$ and repelling for $\lambda \in (\lambda_1, 1]$. **[3]**

(c) Show that if $\lambda \in (\lambda_1, 1]$ then f_λ has a non-zero fixed point. **[4]**

Henceforth, assume that for $\lambda \in (\lambda_1, 1]$ the non-zero fixed point of f_λ is unique, and denoted by x_λ.

(d) Determine the value of λ such that $x_\lambda = 1/6$. **[2]**

(e) Determine the value of λ such that $x_\lambda = 1/2$. **[2]**

(f) Show that if $\lambda = 4\sqrt{3}/9$ then $x_\lambda = 2/3$. **[2]**

(g) Show that the point 1/6 is eventually periodic for the map $f_1(x) = \sin(\pi x)$. **[2]**

(h) Sketch the graph of f_1^3, taking care to mark the value of this function at the points α, β, $1 - \alpha$, $1 - \beta$, where $\alpha = \frac{1}{\pi} \arcsin(1/6)$, $\beta = \frac{1}{\pi} \arcsin(5/6)$. **[3]**

(i) Show that f_1 has a point of least period 3. **[4]**

(j) Determine, with justification, whether f_1 has a point of least period 314159. **[3]**

Question 3 [25 marks]. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 5/6 & \text{if } x < 0 \\ x + 5/6 & \text{if } 0 \leq x < 1/6 \\ 4/3 - 2x & \text{if } 1/6 \leq x < 2/3 \\ x - 2/3 & \text{if } 2/3 \leq x \leq 1 \\ 1/3 & \text{if } x > 1\,. \end{cases}$$

(a) Sketch the graph of f. **[3]**

(b) Determine the fixed point p of f. **[3]**

(c) Determine the orbit under f of the point 0. **[3]**

(d) Show that if $x \in (1/3, 2/3)$, with $x \neq p$, then there exists $N \in \mathbb{N}$ such that $f^N(x) \notin (1/3, 2/3)$. **[6]**

(e) Show that if $x \in [0, 1/3]$ then $f(x) \in [2/3, 1]$, and that if $x \in [2/3, 1]$ then $f(x) \in [0, 1/3]$. Use this to deduce that, for all integers $n \geq 0$, if $x \in [0, 1/3]$ then $f^{2n+1}(x) \in [2/3, 1]$, and that if $x \in [2/3, 1]$ then $f^{2n+1}(x) \in [0, 1/3]$. **[5]**

(f) Using (c), (d), and (e), or otherwise, determine the set of $n \in \mathbb{N}$ such that f has an n-cycle. **[5]**

Question 4 [24 marks].

(a) Given an iterated function system in $\mathbb{R}^2$ defined by the 4 maps $\phi_0(x, y) = (x/3, y/3)$, $\phi_1(x, y) = ((x + 2)/3, y/3)$, $\phi_2(x, y) = (x/3, (y + 2)/3)$, $\phi_3(x, y) = ((x + 2)/3, (y + 2)/3)$, define $\Phi(A) = \cup_{i=0}^{3} \phi_i(A)$ for all sets $A \subset \mathbb{R}^2$, and let F_k denote $\Phi^k([0, 1]^2)$ for $k \geq 0$.

(i) Determine the set F_1. **[3]**

(ii) If F_k is expressed as a disjoint union of N_k closed squares, compute the number N_k. **[3]**

(iii) What is the common side length of each of the N_k squares whose disjoint union equals F_k? **[3]**

(iv) Compute the box dimension of $F = \cap_{k=0}^{\infty} F_k$, being careful to justify your answer. **[5]**

(b) If $C \subset [0, 1]$ denotes the middle third Cantor set, compute the box dimension of the set $C \times [0, 1] = \{(x, y) : x \in C, y \in [0, 1]\} \subset \mathbb{R}^2$. **[5]**

(c) (i) For a map $f : [0, 1] \to \mathbb{R}$, how is the **set of non-escaping points** defined? **[2]**

(ii) Give an example, with justification, of a map f whose set of non-escaping points has box dimension strictly smaller than 1/2. **[3]**

End of Paper.