Queen Mary
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Group Theory

Week 9, Lecture 1, 2 & 3

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Group Actions

G_1, G_2, G_3, G_4 $\quad \pi : G \times X \longrightarrow X$

Definition: Suppose G is a group and X is a set. An action of G on X is a collection $\pi = (\pi_g \,|g \in G)$ of functions from X to X such that:

$\pi : g \times x = \pi_g(x) = gx$

1. ✓ $\pi_1 = \mathrm{id}_X$, and
2. ✓ $\pi_f \circ \pi_g = \pi_{fg}$ for all $f, g \in G$.

$\pi_1(x) = 1 \cdot x$

$\pi_g(x) = g \cdot x$

$\pi_f, \ \pi_g \quad \pi_f \circ \pi_g(x) =$

$= \pi_{f \circ g}(x)$

$\left.\begin{array}{l} g \times x \mapsto y \\ g \times x \to x_1 \end{array}\right\}$

Examples: Trivial Action

$\pi_g(x) = x$

$g \times x \longmapsto x$

$g \times y \longmapsto y$

Group Actions

Lemma: If π is an action of G on X, then each π_g is a permutation on X.

Proof: $g \in G$, $\qquad \pi_g \pi_{g^{-1}} = \pi_{g \circ g^{-1}} = \pi_1 = \pi_e = id_X$

$$\pi_{g^{-1}} \pi_g = id_X,$$

so π_g has inverse so actually

$\pi_g : X \to X$ is an inverse permutation

$$\pi : G \times X \longrightarrow X \qquad \begin{aligned} G &\longrightarrow Sym(X) \\ g &\longmapsto \pi_g \end{aligned}$$

Group Actions

Examples

1. For any X, and a group G we have the trivial action $g.x = x$ for all x ✓
2. S_X symmetric group acting on X. Key example. ✓
3. $GL_n(R)$ acting on R^n ✓
4. $GL(V)$ acting on V ✓
5. G group; Aut(G) acting on G.

If $G \leq S_n$, $X = \{1, 2, \dots n\}$

Then the natural action of G on X

$\pi_g(x) = g(x)$

$\pi_{(123)}(2) = 3$ $\pi_{(312)}(2) = 3$

Group Actions

Examples

For any G, we have conjugate action of G on itself $\tau_g(x) = gxg^{-1}$

$$G \times G \longrightarrow G$$

Group Actions

Action by conjugation

Special case: There is another action on G on itself, but it's not the regular action!

Definition: For $g \in G, x \in G$ set $g_x = gxg^{-1}$. Set $\gamma_g(x) = gxg^{-1}$.
This is a group action of G on itself, and it is an action by automorphisms: $\gamma_g \in Aut(G)$. We say "x is conjugate to y" if there is $g \in G$ such that $g_x = y$.
This is an equivalence relation.
The equivalence classes are called conjugacy classes. Write G/X for the set of equivalence classes. The class of e is $\{e\}$. More generally, the class of x is $\{x\}$ iff $x \in Z(G)$.

Special Remarks: Why is conjugacy important? Because
(1) The action is by automorphisms, so conjugate elements have identical group-theoretic properties (same order, conjugate centralizers etc).
(2) These automorphisms are readily available.
In fact, the map $g \mapsto \gamma_g$ is a group homomorphism $G \to Aut(G)$

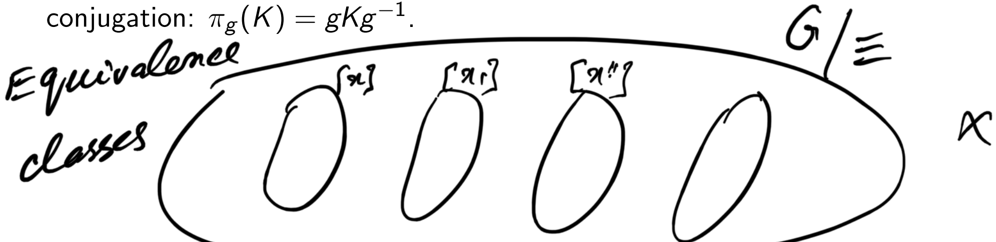
Group Actions

The image of this homomorphism is denoted Inn(G) and called the group of inner automorphisms. The kernel is exactly $Z(G)$, so by first isomorphism theorem $Inn(G) \cong G/Z(G)$. Also, if $f \in Aut(G)$ then $f \circ \gamma_g \circ f^{-1} = \gamma_{f(g)}$.
So Inn(G) $\trianglelefteq$Aut(G).

Example: Aut($\mathbb{Z}^d$) $\cong GL_d(\mathbb{Z})$ but all inner automorphisms are trivial (the group is commutative).

Example: On the other hand, if $|X| \geq 3$ then Inn$(S_X) = S_X$ (the center is trivial).
Out(S_n) =$\{e\}$ except that Out$(S_6) \cong C_2$.

Example: For any G, let X be the set of all subgroups of G. Then G acts on X by conjugation: $\pi_g(K) = gKg^{-1}$.



Group Actions

$\pi_g(x) = g(x) = y \qquad x \equiv y$

Suppose we have an action π of G on X. We define a relation $\equiv$ on X by saying that $x \equiv y$ if there is some $g \in G$ such that $y = \pi_g(x)$.

Lemma: $\equiv$ is an equiavalence relation.

Proof:

Reflexive.

$\pi_1(x) = x$ for all $x \qquad x \equiv x$

Symmetric: Suppose $x \equiv y$, $\exists\, g \in G$

$\pi_g(x) = g(x) = y$

$\pi_{g^{-1}}(y) = \pi_{g^{-1}}(\pi_g(x)) = x$

$\Rightarrow \quad y \equiv x$

Group Actions

Transitive: $x \equiv y$ and $y \equiv z$

$f, g \in G$ $\quad \pi_g(x) = y$ & $\pi_f(y) = z$

$\Rightarrow fg \in G$

$$\pi_{fg}(x) = \pi_f(\pi_g(x)) = \pi_f(y) = z$$

$$x \equiv z$$

Orbits and stabilisers

Definition: Suppose π is an action of a group G on a set X. The **orbit** of π are the equivalence classes under the relation $\equiv$ described above. Given $x \in X$, we write $\underset{\sim}{Orb(x)}$ for the orbit containing x, i.e.

$$\mathrm{Orb}(x) = \left\{ \pi_g(x) \,|\, g \in G \right\}$$

The action π is **transitive** if there is only one orbit. Write G/X for the set of orbits.

Definition: Suppose π is an action of a group G on a set X, and let $x \in X$. The **stabiliser** of x is the set

$$\mathrm{Stab}(x) = \left\{ g \in G \,|\, \pi_g(x) = x \right\}.$$

$g_1 x = x$ $\quad g_3 x = x$

$g_2 x = x$

Example: $\pi : G \times X \longrightarrow X$, $\quad \pi_g(x) = g(x) = x$

$\mathrm{Orb}(x) = \{ \{x\} \}$ $\qquad \mathrm{Stab}(x) = G$

Orbits and stabilisers

$G \times X \to X$ $\quad \pi_g : X \to X \quad g(x) = y$

Lemma

Suppose π is an action of G on X, and let $x \in X$. Then $\mathrm{Stab}(x) \leq G$.

$Stab(x) = \{ g \mid g(x) = x \}$

$\pi_g(x) = x$

Proof:

$\pi_1(x) = x$

$\Rightarrow \quad 1 \in Stab(x) \neq \{\emptyset\}$

Suppose $f, g \in G$, $\quad \pi_f(x) = x, \quad \pi_g(x) = x$

$\pi_f \; \pi_g \in Stab(x)$

$\pi_{fg^{-1}}(x) = \pi_f(\pi_{g^{-1}}(x)) = \pi_f(\pi_{g^{-1}}(\pi_g(x)))$

$= \pi_f(x) = x$

So $fg^{-1} \in Stab(x)$

Hence a Subgroup.

Orbits and stabilisers

Examples: Let's work out some examples of orbits and stabilisers. (i) Take any G and X, and let π be the trivial action. Then for any $x \in X$,

$$\mathrm{Orb}(x) = \{x\}, \qquad \mathrm{Stab}(x) = G.$$

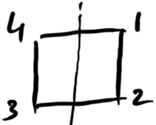
- Take $G = \mathcal{D}_8$, and let X be the set of vertices of the square, numbered $1, 2, 3, 4$ in clockwise order starting from the top right. Then G acts on X in a natural way: $\pi_g(x) = g(x)$. Taking $x = 1$, we get

$$\mathrm{Orb}(1) = \{1, 2, 3, 4\}, \qquad \mathrm{Stab}(1) = \{1, rs\} = \{R_0, R_{13}\}.$$

- Let G be any group, and let π be the regular action of G on G. Then any two elements $g, h \in G$ lie in the same orbit, because $\pi_{hg^{-1}}(g) = h$. So this action is transitive. $\mathrm{Stab}(h) = \{1\}$ for any $x \in G$.

$\mathrm{Orb}(1) = \{1, 2, 3, 4\}$

$G = \mathcal{D}_8 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$



$\pi_r(1) = 2$ $\qquad$ $\pi_{r^3}(1) = 4$

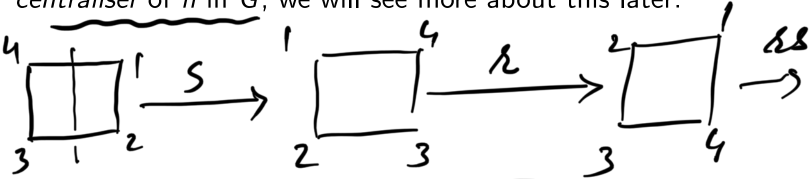
$\pi_{r^2}(1) = 3$ $\qquad$ $\pi_s(1) = 4$

$\mathrm{Stab}(1) = \{1,$

Orbits and stabilisers

$\pi_g(h) = ghg^{-1}$ $\qquad \pi_g : G \to G$

- Let G be any group, and let π be the conjugation action of G on G, and let $h \in G$. Then Orb(h) is just the conjugacy class $ccl(h)$. Stab(h) is called the *centraliser* of h in G; we will see more about this later.



$$\mathrm{Orb}(h) = \{ \underbrace{ghg^{-1}}_{\pi_g(h)} \mid g \in G \}$$

$$\mathrm{Stab}(h) = \{ \underbrace{ghg^{-1} = h}_{\pi_g(h) = h} \mid g \in G \} = \{ gh = hg \mid g \in G \} = Z(G)$$

Orbits and stabilisers

$|G|$

Orbit Stabilizer Theorem

Suppose π is an action of G on X, and $x \in X$. Then $|G| = |Orb(x)||Stab(x)|$.

Alternatively

There is a bijection between the orbit $O(x) \subseteq X$ and $G/Stab_G(x)$. Moreover, the stabilizers of an orbit of G is a conjugacy class ~~in~~ of subgroups

Proof:

$Stab(x) \leq G$ $\qquad$ $|G| = \boxed{|G : Stab(x)|}\,|Stab$

where $|G : Stab(x)|$ is the number of left cosets of $Stab(x)$ in G.

$Orb(x) = \{\pi_g(x) \mid g \in G\}$

$\theta : \quad g\,Stab(x) \longmapsto \pi_g(x)$ ✓

Orbits and stabilisers

$$f\,\underline{Stab(x)} = g\,\underline{Stab(x)}$$ by coset Lemma.

$f^{-1}g \in Stab(x)$, $\quad \lambda_{f^{-1}g}(x) = x$

$$\lambda_f(x) = \lambda_f(\lambda_{f^{-1}g}(x)) = \lambda_{ff^{-1}}(\lambda_g(x)) = \lambda_g(x)$$

<u>Injective - one-to-one.</u>

$\theta(f\,Stab(x)) = \theta(g\,Stab(x))$ i.e $\lambda_f\underline{(x) = }\lambda_g(x)$

$$\lambda_{f^{-1}g}(x) = \lambda_{f^{-1}}(\lambda_g(x)) = \lambda_{f^{-1}}(\lambda_f(x)) = x$$

$\Rightarrow f^{-1}g \in Stab(x)$, so by coset Lemma
$$f\,Stab(x) = g\,Stab(x)$$

Orbits and stabilisers

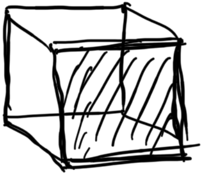
Surjective: $g \in G$, $\pi_g(x)$, $\pi_g(x) = \theta(g\,Stab(x))$

We can apply the Orbit–Stabiliser Theorem to find the size of a group when we have an action that we understand.

$G \times X$

Example: Let G be the symmetry group of a cube. Let x be a face of the cube, and let G act on the set of faces. This is a transitive action, because (it's easy to see that) you can get from any face to any other by applying a symmetry of the cube. So $|Orb(x)| = 6$. Now think about $Stab(x)$. Notice that any symmetry of the cube which fixes x gives a symmetry of x: a rotation of the cube gives a rotation of x, and a reflection of the cube gives a reflection of x. Conversely, any symmetry of x can be extended to a symmetry of the whole cube. So $Stab(x)$ is isomorphic to the symmetry group of x, which is $\mathcal{D}_8$. In particular, $|Stab(x)| = 8$. So by the Orbit-Stabiliser Theorem, $|G| = 6 \times 8 = 48$.

Application of Orbit-Stabiliser Theorem



Orbits and stabilisers

$S_8 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$

Example: $S = \{1,2,3,4,5,6,7,8\}$. The group of permutation on the set S is

$$G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}$$

$G \times S \to S$

$G \times S \mapsto S_8$

$\mathrm{Stab}_G(1) = \{(1), (78)\}$
$\mathrm{Orb}_G(1) = \{1, 3, 2\}$
$\mathrm{Stab}_G(2) = \{(1), (78)\}$
$\mathrm{Orb}_G(2) = \{2, 1, 3\}$
$\mathrm{Stab}_G(7) = \{(1), (123)(456), (132)(465)\}$
$\mathrm{Orb}_G(7) = \{7, 8\}$

$$(1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$(132)(465)(78) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 6 & 4 & 5 & 8 & 7 \end{pmatrix}$$

$$(78) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}$$

Orbits and stabilisers

$D_8 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$

Example: $G = D_8$ set of all geomteries of square.

$C_{D_8}(rs) = \{1, r^2, rs, r^3s\}$

8 2 4

$|D_8| = |ccl(rs)||C_G(rs)|$

by orbit-stabilizer Theorem.

Centaliser and normaliser

Definition: Suppose G is a group.

- If $h \in G$, the **centraliser** of h is

$$C_G h = \{g \in G \mid gh = hg\}.$$

- If $H \leq G$, the **normaliser** of H is

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}.$$

$$N_g(h) = \{ g \in G \mid ghg^{-1} = h \}$$

Orbits and stabilisers

Proposotion

Suppose G is a group, $h \in G$ and $H \leq G$. Then $h \in C_G(h) \leq G$, and $H \trianglelefteq G \trianglelefteq N_G(H) \leq G$.

$$C_G(h) = \{ g \in G \mid gh = hg \} = \{ g \in G \mid ghg^{-1} = h \}$$

So $C_G(h)$ is the stabilizer of h under the conjugation action of G on G.

$$C_G(h) \leq G.$$

$N_G(H)$ is the stabilizer of H under the conjugation

Orbit Counting Lemma

$N_G(h) \leq G$

For each $h \in H$, we have

$$hHh^{-1} = \{ hbh^{-1} \mid b \in H \} = H$$

so $H \leq N_G(H)$,

$g \in N_G(H)$ $\quad gHg^{-1} = H$ so

$$H \trianglelefteq N_G(H)$$

Exams Style Questions

Example: $\mathbb{F}$, $\underline{G = GL_2(\mathbb{F})}$, $\check{X} = \mathbb{F}^2$, $\begin{pmatrix} a \\ b \end{pmatrix}$, the set of column vectors of length 2.

$$\pi_{\underline{g}}(\underline{x}) = g\,x$$

Exams Style Questions

Exercise. Let $G = GL(2,\mathbb{R})$ and $X = \mathbb{R}^2$.

(1) Show that the map

$$G \times X \to X, \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},$$

defines a G action.

(2) What are the orbits and fixed point sets of this G action?

$\pi_g : X \to X$

$\pi_g(x) = g(x)$

Defines an action when

The collection of all invertible matrices constitutes the general linear group $GL(2,\mathbb{R})$.

$\pi_1(x) = 1.x = x$

$\pi_g(\pi_f(x)) = \pi_{gf}(x)$

Identity & closure

Exams Style Questions

Exams Style Questions

Exercise. Let $H \leq G$, and define H action by restricting the map $H \times X$. Calculate the orbits and fixed point sets in the following cases:

(1) $H = SO(2)$.

(2) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(3) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_{>0} \right\}$.

(4) $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$.

(5) $H = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$.

Exams Style Questions

QMplus Quiz

Attempt Quiz 9 at QMplus page

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$