Selected solutions to PS 8.

1. (1) $\Delta(\ln r)$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\ln r)$$

$$= \left(\frac{\partial^2}{\partial r^2}\right)\ln r + \frac{1}{r}\frac{\partial}{\partial r}(\ln r) + 0$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + r \cdot \frac{1}{r} + 0$$

$$= \frac{-1}{r^2} + \frac{1}{r^2} = 0$$

So harmonic

1. (3) $\Delta\left(r^2 \cos 2\theta\right)$

$$= \frac{\partial^2}{\partial r^2}(r^2\cos 2\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r^2 \cos 2\theta) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^2 \cos 2\theta)$$

$$= \frac{\partial}{\partial r}(2r\cos 2\theta) + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}\left[r^2(-4\cos 2\theta)\right]$$

$$= 2\cdot\cos 2\theta + 2\cdot\cos 2\theta - 4\cos 2\theta$$

$$= 0$$

So also harmonic !

2. $\Delta v(x,y)$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[u(\lambda x, -\lambda y)\right]$$

$$= \frac{\partial^2}{\partial x^2}\left[u(\lambda x, -\lambda y)\right] + \frac{\partial^2}{\partial y^2}\left[u(\lambda x, -\lambda y)\right]$$

$$= \frac{\partial}{\partial x}\left[\lambda u_x(\lambda x, -\lambda y)\right] + \frac{\partial}{\partial y}\left[-\lambda u_y(\lambda x, -\lambda y)\right]$$

$$= \lambda^2 u_{xx}(\lambda x, -\lambda y) + (-\lambda)^2 u_{yy}[\lambda x, -\lambda y)$$

$$= \lambda^2 \left(u_{xx} + u_{yy}\right)$$

$$= 0$$

so v is also harmonic.

also. for the domain of definitions.
we have u defined on $B_1(0)$,

so $(\lambda x, -\lambda y) \in B_1(0)$

Namely $\lambda^2 x^2 + \lambda^2 y^2 \leq 1$

$$x^2 + y^2 \leq \frac{1}{\lambda}$$

in other words

$$(x, y) \in B_{\frac{1}{\lambda}}(0)$$

v is defined on the disk $B_{\frac{1}{\lambda}}(0)$.

5 (1) Using the first mean value theorem

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} \left[3\sin\theta - 4\cos 4\theta + 1\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 3\sin\theta \, d\theta - \frac{1}{2\pi}\int 4\cos 4\theta \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} 1 \, d\theta$$

$$= 0 + 0 + 1$$

$$= 1$$

(2) By the maximum principle, The Max / min happens on the bourdary.

Noticing $-1 \leq \cos\theta, \sin\theta \leq 1$, we know on the bourdary thay

$$u(2,\theta) = 3\sin\theta - 4\cos 4\theta + 1$$

$$\leq 3 + 4 + 1$$

$$= 8$$

and

$$u(2,\theta) = 3\sin\theta - 4\cos 4\theta + 1$$

$$\leq -3 - 4 - 1$$

$$= -6$$

So $-6 \leq u \leq 8$ on the whole Ω

8. We have deduced that the general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2} k t} \sin \frac{n\pi x}{L} \quad \text{with } L=1$$

in this question

Thus $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\pi^2 n^2 k t} \sin(n\pi x)$.

The initial condition gives $-6 \sin(6\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$

Using the orthogonalities of $\sin(n\pi x)$, we "observe" that $a_n = 0$ except for $n = 6$

And $a_6 = -6$. (the $n=6$ term need to match)

So the solution to this Dirichlet problem is

$$u(x,t) = -6 e^{-36\pi^2 k t} \sin(6\pi x)$$

$u \to 0$ as $t \to \infty$.

9. Using separation of variables, suppose

$$u(x,t) = X(x) T(t).$$

plug into equation get

$$X \dot{T} = k X'' T$$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} = -\lambda \quad \text{a constant.}$$

We get 2 ODEs

$$\begin{cases} X'' = -\lambda X \\ \dot{T} = -\lambda k T \end{cases}$$

The 1st ODE and the boundary condition gives an eigenvalue problem.

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0, \ X'(L) = 0 \end{cases}$$

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0, \ X'(L) = 0 \end{cases}$$

using integration by parts, we have shown in problem set 5 Question 2 the same eigenvalue problem that $\underline{\lambda \geq 0}$.

so $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

and $X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

The first bondary condition gives

$$0 = X'(0) = B\sqrt{\lambda}, \text{ so } B = 0 \text{ and } A \neq 0$$

the second bondary condition gives

$$0 = X'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot L)$$

we have $\sqrt{\lambda}\cdot L = n\pi$, $n = 1, 2, \cdots$

so the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$

The eigenfunctions are

$$X_n(x) = \cos\frac{n\pi x}{L}$$

knowing λ_n, we solve $\dot{T} = -k\lambda_n T$ and get

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}$$

For $n=0$, i.e. $\lambda = 0$, we get

$$X'' = 0, \quad X(x) = a_0 + b_0 t$$

the bondary condition $X'(0) = X'(L) = 0$ gives $b_0 = 0$

so $X_0(x) = \text{Const}$, solving for $\dot{T} = 0$ we get

$$T_0(t) = \text{Const}$$

The general solutions are

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n X_n(x) T_n(t)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}kt} \cos\frac{n\pi x}{L} .$$

10. (1) plug in $t=0$, we get

$$1 = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

using that $\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$ $(*)$

we have (by multiply both sides by $\sin\frac{m\pi x}{L}$)

$$\int_0^L \cos\frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} = a_m \cdot \frac{L}{2}$$

we get $a_m = \frac{2}{L} \int_0^L \cos\frac{m\pi x}{L}$

$$= \frac{2}{L} \frac{L}{m\pi} \left[\sin\frac{m\pi x}{L}\right]\Big|_0^L$$

$$= \frac{2}{L} \frac{L}{m\pi} [0-0]$$

$$= 0 \quad \text{for } m \geq 1$$

so the initial condition gives $1 = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = a_0$

The solution is

$$u(x,t) \equiv 1$$

10(2) Notice $\cos^2\left(\frac{\pi x}{L}\right) = \frac{\cos\frac{2\pi x}{L}+1}{2}$

so $\frac{1}{2}+\frac{1}{2}\cos\frac{2\pi x}{L} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

we can use ther orthogonality condition (**)

and "observe" that

$a_0=\frac{1}{2}$, $a_2=\frac{1}{2}$, $a_n=0$ for $n\neq 0,2$

so plug into the general solutions obtained in 5 and get

$u(x,t) = \frac{1}{2}+\frac{1}{2}e^{-\frac{4\pi^2}{L^2}kt}\cos\frac{2\pi x}{L}$

11 :

separation of variables gives $u(x,t)=X(x)T(t)$

and $X\dot{T} = kX''T$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} = -\lambda$$

with $\lambda\geq 0$. (proof of $\lambda\geq 0$ is similar to the previous problems using integration by parts)

The eigenvalue problem is then

$$\begin{cases} X''=-\lambda X \\ X(-L)=X(L) \\ X'(-L)=X'(L) \end{cases}$$

The general solutions for $X(x)$ is

$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

and so

$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

using $X(-L)=X(L)$, we have

$$C_1\cos(-\sqrt{\lambda}L)+C_2\sin(-\sqrt{\lambda}L)=C_1\cos(\sqrt{\lambda}L)+C_2\sin(\sqrt{\lambda}L)$$

$$C_1[\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)]=C_2[\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)] \quad ①$$

using $X'(-L)=X'(L)$, we have

$$-C_1\sqrt{\lambda}\sin(-\sqrt{\lambda}L)+C_2\sqrt{\lambda}\cos(-\sqrt{\lambda}L)=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)+C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

$$C_2\sqrt{\lambda}[\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)]=C_1\sqrt{\lambda}[\sin(-\sqrt{\lambda}L)-\sin(\sqrt{\lambda}L)] \quad ②$$

There are 2 cases:

Case 1:

If $\lambda=0$, then we have $X(x)=\text{const}$, $\lambda_0=0$

Case 2:

If $\lambda\neq0$, we have 2 sub cases:

sub case 2.1: If neither $[\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)]$ nor $[\cos(\sqrt{\lambda}L)-\cos(-\sqrt{\lambda}L)]$ is zero,

by ①, $$\frac{C_1}{C_2}=\frac{\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)}{\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)}$$

by ②, $$-\frac{C_2}{C_1}=\frac{\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)}{\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)}$$

so $$\frac{C_1}{C_2}=\frac{-C_2}{C_1}$$

multiply both sides by a_1a_2, get

$$c_1^2 = -c_2^2$$
$$c_1^2 + c_2^2 = 0$$

So $c_1 = 0$ and $c_2 = 0$, this is the trivial solution $X \equiv 0$.

Sub case 2.1: either $\sin(\sqrt{\lambda}L) - \sin(-\sqrt{\lambda}L) = 0$

or $\cos(\sqrt{\lambda}L) - \cos(-\sqrt{\lambda}L) = 0$,

If $\sin(\sqrt{\lambda}L) - \sin(-\sqrt{\lambda}L) = 0$

then $\sqrt{\lambda}L - (-\sqrt{\lambda}L) = 2n\pi, \quad n = 1, 2, \cdots$

$$2\sqrt{\lambda}L = 2n\pi$$
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots$$

If $\cos(\sqrt{\lambda}L) - \cos(-\sqrt{\lambda}L) = 0$

then $\sqrt{\lambda}L - (-\sqrt{\lambda}L) = 2n\pi, \quad n = 1, 2, \cdots$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

both give $X_n(x) = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$.

Knowing λ_n, we have

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}$$

The general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

Using the initial condition, we have
when $t=0$

$$f(x)=u(x,0)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right]$$

By the formula of Fourier series coefficients,
we get

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

12. Notice

$$f(x)=\sin^2\left(\frac{\pi x}{L}\right)=\frac{1-\cos\frac{2\pi x}{L}}{2}$$

$$=\frac{1}{2}-\frac{1}{2}\cos\frac{2\pi x}{L}$$

By the orthogonality of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$,
we know that

$a_0 = \frac{1}{2}$,

$a_n = 0$ except for $n=2$

$a_2 = -\frac{1}{2}$

$b_n = 0$ for all n.

so plugging into the general solutions, we get.

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{L}\, e^{-\frac{4\pi^2}{L^2}kt}$$