To justify the existence of such a dense orbit, we work symbolically, and let

$$s = \underbrace{0\ \ 1}_{\text{length-1 blocks}}\ \ \underbrace{00\ \ 01\ \ 10\ \ 11}_{\text{length-2 blocks}}\ \ \underbrace{000\ \ 001\ \ 010\ \ 011\ \ 100\ \ 101\ \ 110\ \ 111}_{\text{length-3 blocks}}\ \ldots$$

where we build up the sequence s by first writing down all length-1 'blocks'/'words',

then " " " " 2 " ,

" " " " " 3 "

" " " " " 4 "

⋮

" " " length-n "

Define $x_0 = h(s)$

$$= \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \cdots$$

We claim that x_0 has a dense orbit

Now choose some (other) $x \in [0,1)$, and suppose x has binary expansion

$$b(x) = b_1 b_2 b_3 b_4 \ldots$$

(i.e. ~~$h(x) =$~~ $h(b_1 b_2 b_3 b_4 \ldots) = x$)

Then it is possible (due to the recipe for defining s) to choose some $i \in \mathbb{N}$ such that $\sigma^i(s)$ and $b(x)$ begin with a common (long) block of digits/symbols i.e. $\sigma^i(s) = \underbrace{b_1 b_2 \dots b_k}_{\text{Same symbols as } b(x) \text{ begins with}} c_{k+1} c_{k+2} \dots$

This ensures that

$$|D^i(x_0) - x| < \frac{1}{2^k} < \varepsilon \quad \text{if}$$

k is chosen sufficiently large.

So x_0 does have a dense orbit under D.

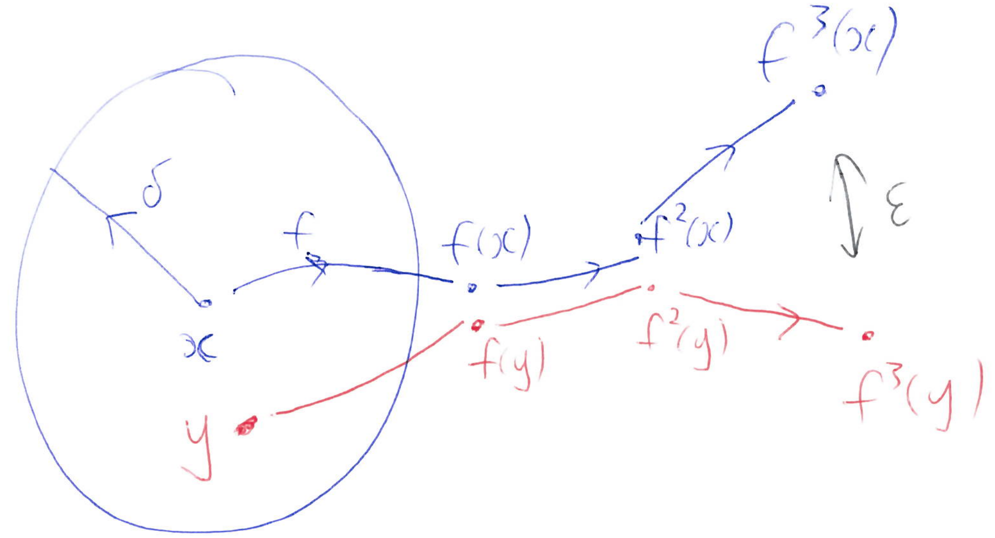
The tent map T, and the logistic map f_4 both also have the property that there exist points whose orbits are dense.

Although there is no agreed definition of 'chaos', there is some universal agreement that an indication of chaotic dynamics is <u>sensitive dependence on initial conditions</u> (SDIC):

<u>Defn</u> Let $I \subset \mathbb{R}$ be an interval, and consider $f: I \to I$. We say that f has <u>sensitive dependence on initial conditions</u> (SDIC) at $x \in I$ if there $\varepsilon > 0$ such that for all $\delta > 0$ there exists $y \in I$ with $|x - y| < \delta$, <u>and</u> there exists $n \in \mathbb{N}$ such that

$$|f^n(x) - f^n(y)| > \varepsilon.$$

We say that $f: I \to I$ has SDIC if it has SDIC at every $x \in I$.



i.e. There are points arbitrarily close to x whose orbits diverge/separate from the orbit of x (by at least ε)

<u>Proposition</u> The doubling map $D : [0,1) \circlearrowleft$ (given by $D(x) = 2x \pmod 1$) has SDIC at <u>every</u> $x \in [0,1)$.

i.e. The map D has SDIC.

<u>Proof</u> Let $\varepsilon = 1/3$. Suppose $x \in [0,1)$. Given $\delta > 0$, choose $n \in \mathbb{N}$ such that

$\frac{1}{2^{n+1}} < \delta$, and let

$$y = \begin{cases} x + \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [0, \frac{1}{2}) \\ x - \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [\frac{1}{2}, 1) \end{cases}$$

Then $|y - x| = \frac{1}{2^{n+1}} < \delta$,
and

$$|D^n(y) - D^n(x)| = \frac{1}{2} > \frac{1}{3} = \varepsilon. \qquad \square$$

It can also be shown that:

<u>Prop</u> The tent map $T: [0,1] \circlearrowleft$
(given by $T(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$)
has SDIC

Lemma If $f: I \to I$ has SDIC, and is topologically conjugate to some $g: I \to I$, then g has SDIC.

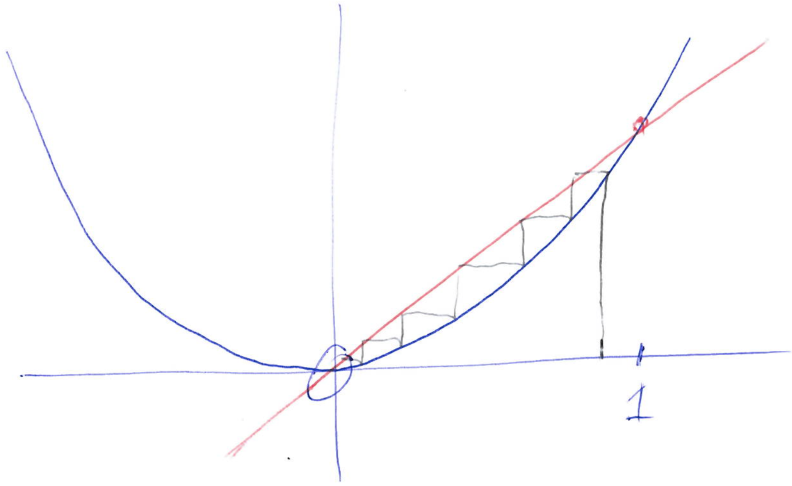
Consequently:

Prop The logistic map $f_4 : [0,1] \to [0,1]$ (given by $f_4(x) = 4x(1-x)$) has SDIC.

Remark

(a) A map $f: \mathbb{R} \to \mathbb{R}$ of the form $f(x) = x + c$ $(c \in \mathbb{R})$ does *not* have SDIC at *any* point
(if $|x-y| = \delta$ then $|f^n(x) - f^n(y)| = \delta$ for all $n \geq 0$)

(b) The map $f(x) = x^2$ does <u>not</u> have SDIC at any point $x \in (-1,1)$



Recall that 0 is an attracting fixed point, and $(-1,1)$ is its basin of attraction.

Definition (from Robert Devaney's book "An Introduction to Chaotic Dynamical Systems")

Let $I \subset \mathbb{R}$ be an interval. We say that $f : I \to I$ is chaotic (in the sense of Devaney) if

(i) f has SDIC (at all points in I)

(ii) the set of all periodic points of f is a dense subset of I

(iii) there is an orbit (under f) of some point which is a dense subset of I.

This definition of Devaney dates back to the mid-1980s

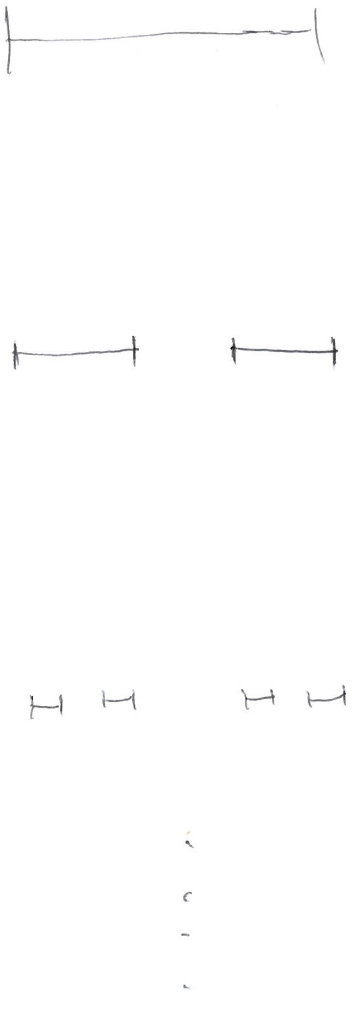
It was later shown (in the mid-1990s) that if $f: I \to I$ is <u>continuous</u> then property (iii) implies (i) and (ii)

So the definition of 'chaotic' (in the sense of Devaney) can be reduced to saying that some point has a dense orbit under f (<u>if</u> f is continuous).

<u>Examples</u> The following maps are chaotic in the sense of Devaney:

- The doubling map $D: [0,1) \to [0,1)$
- The tent map $T: I \to I$
- The logistic map $f_4(x) = 4x(1-x)$
  $f_4 : I \to I$

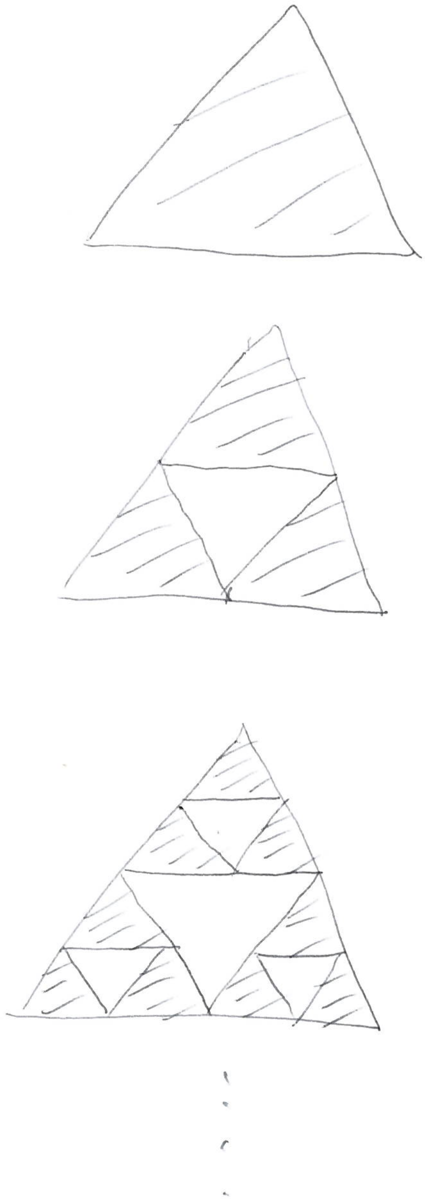
Fractals — Pictorial motivation



Step 0

Step 1

Step 2



"Middle-Third Cantor Set"

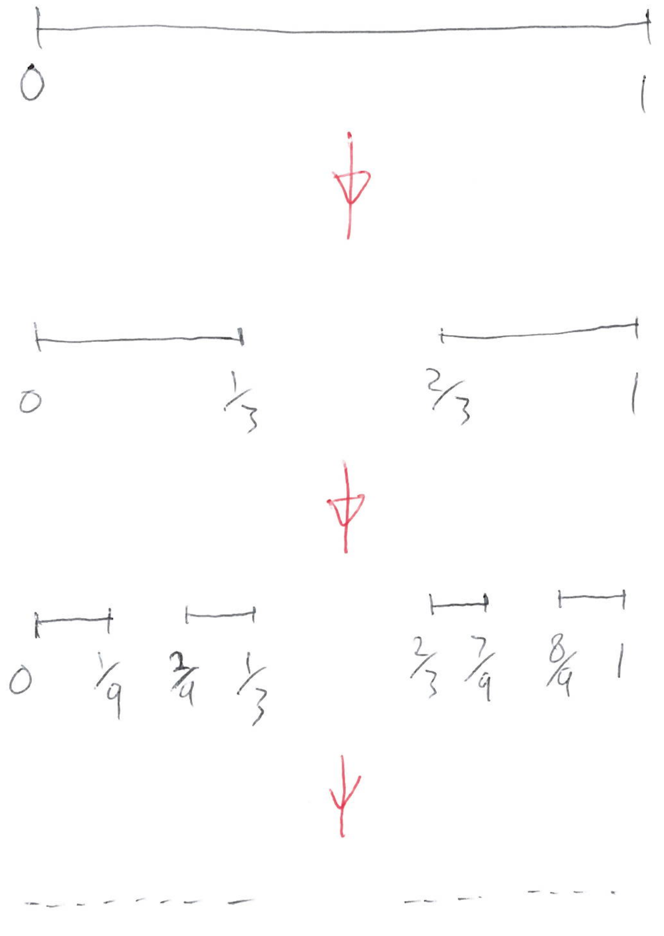
Sierpinski Triangle

These objects are obtained "in the limit" as the number of steps tends to ∞

(middle-third) Cantor set

We ~~m~~ shall now construct, in more detail, the <u>middle-third Cantor set</u>.

Let $C_0 = [0,1] \subset \mathbb{R}$, and consider the following inductive definition of C_n:



$C_0 = [0,1]$

(Remove the open 'middle third' $(1/3, 2/3)$ of this interval)

$C_1 = [0, 1/3] \cup [2/3, 1]$

Remove the open middle third of the 2 disjoint intervals making up C_1

$C_2 = [0, 1/9] \cup [2/9, 1/3]$

$\cup \, [2/3, 7/9] \cup [8/9, 1]$

We continue this indefinitely, and at step n:

$$C_n = C_{n-1} \setminus \left\{ \begin{array}{c} \text{open middle thirds} \\ \text{of all sub-intervals} \\ \text{of } C_{n-1} \end{array} \right\}$$

Note that C_n is a union of 2^n disjoint closed intervals of the form

$$\left[\frac{k}{3^n}, \frac{k+1}{3^n} \right] \quad \left(\begin{array}{c} \text{these intervals} \\ \text{have length } 1/3^n \end{array} \right)$$

__Definition__ The __middle third Cantor set__ is

$$C = \bigcap_{n=0}^{\infty} C_n$$

(= the set of points which lie in all of the C_n, i.e. never lie in the open middle third of any sub-interval)

Observe that the total 'length' of C_n is

$$2^n \times (1/3)^n = (2/3)^n,$$

which $\to 0$ as $n \to \infty$, so C has length 0.

A more general notion:

<u>Defn</u> A set of the form $\varphi(C)$, where C is the middle third Cantor set, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, is called a <u>Cantor set</u>.

Cantor sets K have the following properties (which could be regarded as *defining* properties of the notion of Cantor set):

1. K is "perfect", i.e. each point in K has other points in K which are arbitrarily close to it.

2. K is 'totally disconnected', ~~e.g.~~ i.e. K ~~does~~ contains no non-empty open intervals

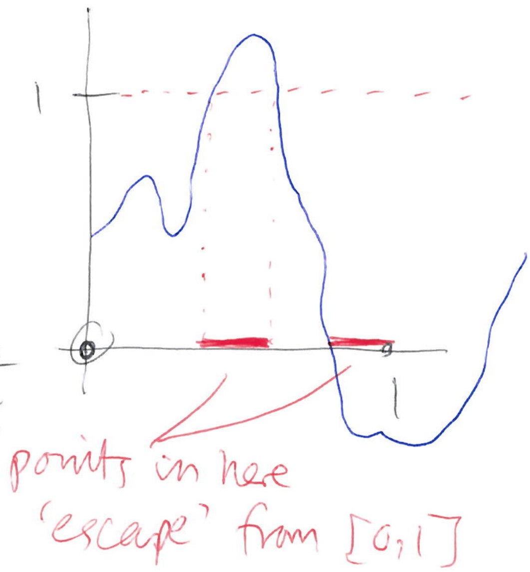
3. K is 'closed' (i.e. every sequence of points in K which converges to a limit l, is such that $l \in K$)

Que How is this discussion related to dynamical systems?

Defn For a map $f : [0,1] \to \mathbb{R}$, point $x \in [0,1]$ is said to be non-escaping if $f^n(x) \in [0,1]$ for all $n \geq 0$.

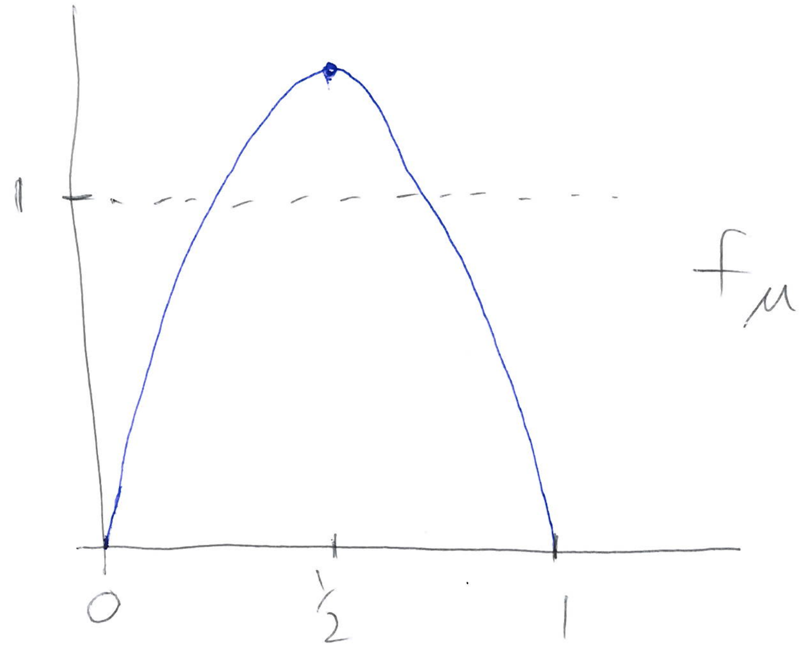
(i.e. the orbit of x does not 'escape' from the domain of definition $[0,1]$)

The set $\Lambda = \Lambda(f)$ $:= \{$non-escaping points$\}$ is called the non-escaping set or the set of non-escaping points.



The non-escaping set of the logistic map $f_\mu : [0,1] \longrightarrow \mathbb{R}$, defined by $f_\mu(x) = \mu x(1-x)$, turns out to be a ~~fractal~~ Cantor set for $\mu > 4$.

(recall, previously we only considered the parameter μ satisfying $0 \leq \mu \leq 4$)



Notice that since $\mu > 4$, the graph of f_μ rises above the line $y = 1$ in a central interval around $x = \frac{1}{2}$.

All points in this interval 'escape' from $I = [0,1]$ after a single iterate of f_μ, and therefore do not belong to the non-escaping set $\Lambda(f_\mu)$.

The non-escaping set $\Lambda(f_\mu)$ is given by

$$\Lambda(f_\mu) = \{x \in [0,1] : f_\mu^n(x) \in [0,1] \ \forall n \geq 0\}$$

$$= \bigcap_{n=0}^{\infty} \Lambda_n$$

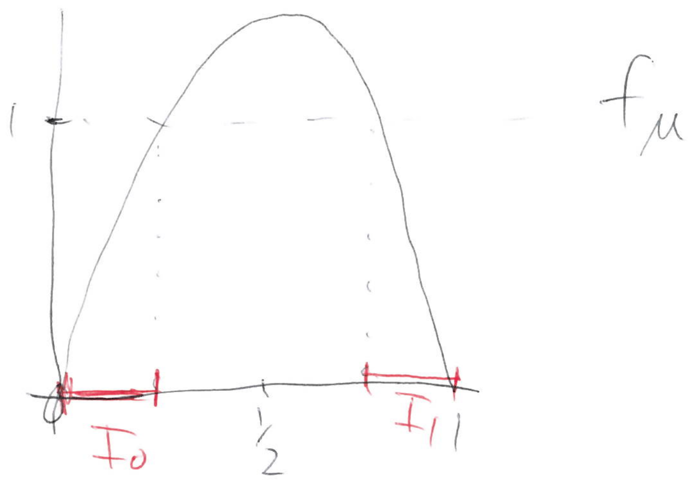
where $\Lambda_n = \{x \in [0,1] : f_\mu^n(x) \in [0,1]\}$

$= \{x \in [0,1] : f_\mu^n(x)$ is defined and belongs to $[0,1]\}$

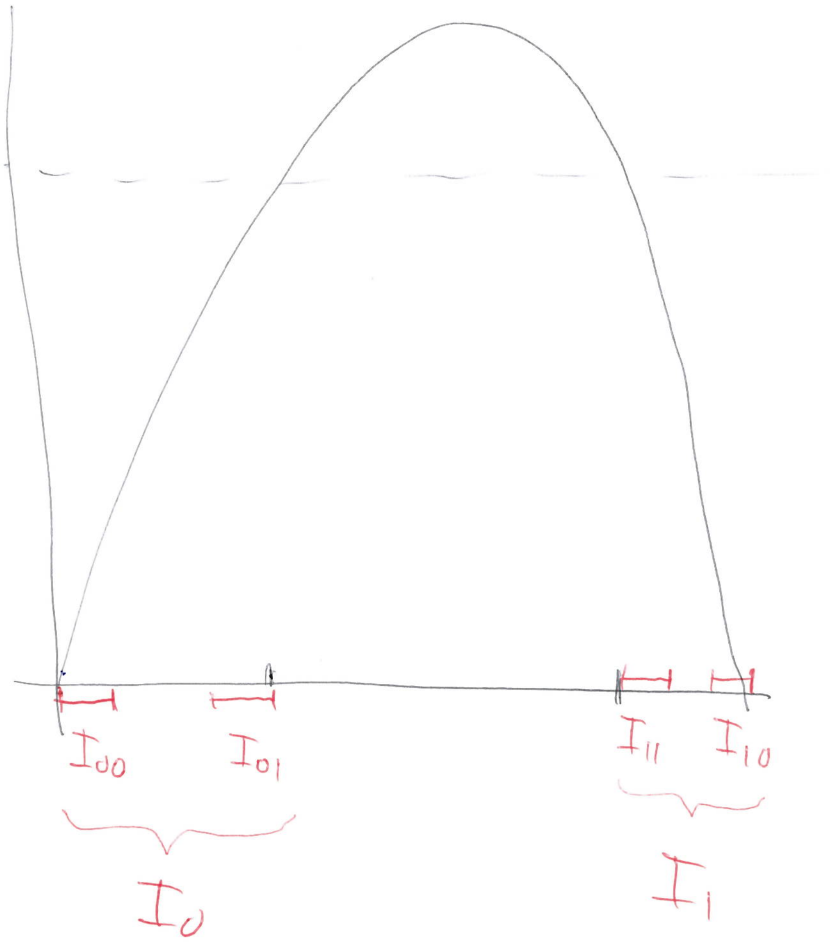
$= \{x \in [0,1] : f_\mu^i(x) \in [0,1]$ for all $0 \leq i \leq n\}$

Note that $\Lambda_0 = [0,1]$

Note that $\Lambda_1 = I_0 \cup I_1$ is the union of two disjoint closed intervals (I_0 and I_1, where $I_0 \subset [0, 1/2)$, and $I_1 \subset [1/2, 1]$)



Note that $\Lambda_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$
is the union of four disjoint closed intervals.



In general, the set Λ_n can be written as a disjoint union of 2^n closed intervals

$$I_{b_1 b_2 \ldots b_n} = \{ x \in [0,1] : f_\mu^{i-1}(x) \in I_{b_i} \text{ for } 1 \le i \le n \}$$

where $b_i \in \{0, 1\}$