PROBLEM SET 8 FOR MTH 6151

1. Verify that the following function expressed in polar coordinates are all harmonic functions, find their domain of definition (the region that the function is well defined), and then write down their expression in Cartesian coordinate.

(1) $\ln r$.
(2) θ.
(3) $r^2 \cos(2\theta)$.
(4) $r \ln r \sin\theta + r\theta\cos\theta$.

2. Suppose $U(x, y)$ is a harmonic function in $B_1(0)$ (disk of radius 1 centered at the origin) and $\lambda > 0$ is a constant, show that $V(x, y) = U(\lambda x, -\lambda y)$ is also a harmonic function, defined on the disk $B_{\frac{1}{\lambda}}(0)$.

3. (1) Find the harmonic function on the annular region

$$\Omega = \{\frac{1}{2} < r < 2\}$$

satisfying the boundary conditions given by

$$U(\frac{1}{2}, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta,$$
$$U(2, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta.$$

(2) Show that this is the unique harmonic function satisfying these boundary conditions.

4. Find the harmonic function on domain obtained by plane digged out a ball

$$\Omega = \{r \geq 1\} = \mathbb{R}^2 \setminus B_1(0)$$

satisfying the boundary conditions given by

$$U(1, \theta) = \cos\theta,$$

that satisfies $U(r, \theta) \to 0$ as $r \to \infty$.

5. Suppose that U is a harmonic function in the disk $\Omega = \{r < 2\}$ and that

$$U(2, \theta) = 3\sin\theta - 4\cos 4\theta + 1.$$

(1) Without finding the solution, compute the value of U at the origin —i.e. at $r = 0$.
(2) Without finding the solution, show that $U \geq -6$ and $U \leq 8$ on the whole disk Ω.

6. Use the maximum principle for the Laplace equation to show that the solution to the Poisson equation

$$\Delta U = f$$

over a domain Ω with boundary condition $U = 0$ on $\partial\Omega$ is unique.

7. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0, \frac{\pi}{2}], \quad t \geq 0, \\ &U(x,0) = \sin x, \\ &U(0,t) = 0, \qquad U_x(\frac{\pi}{2}, t) = 0. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

(Observe that this problem has mixed boundary conditions —Dirichlet to the left and Neumann to the right.)

8. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0, 1], \quad t \geq 0, \\ &U(x,0) = -6\sin(6\pi x), \\ &U(0,t) = 0, \qquad U(1, t) = 0. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

9. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0, L], \quad t \geq 0, \\ &U(x,0) = f(x), \\ &U_x(0,t) = 0, \qquad U_x(L, t) = 0. \end{aligned}$$

Observe that this problem has Neumann boundary conditions.

10. Compute the solution of Exercise 5 if:

(i) $f(x) = 1$,
(ii) $f(x) = \cos^2(\pi x/L)$.

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

11. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [-L, L], \quad t \geq 0, \\ &U(x,0) = f(x), \qquad f(L) = f(-L). \\ &U(-L,t) = U(L,t), \\ &U_x(-L,t) = U_x(L,t). \end{aligned}$$

The boundary conditions used in this problem are called periodic. Can you imagine a physical system described by this problem?

12. Compute the solution in Exercise 7 in the case

$$f(x) = \sin^2(\pi x/L).$$

What is the value of $U(x,t)$ as $t \to \infty$?