WEEK 9 NOTES

1. Uniqueness property of inhomogeneous Laplace equations (Poisson equations)

Using the maximum principle, we can show uniqueness of solution to the Laplace and Poisson equation.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain, then the Poisson equation on Ω*

$$\Delta U = \psi$$
$$U|_{\partial\Omega} = f,$$

has a unique solution

Proof. Suppose there are 2 solutions U_1, U_2, then $V = U_1 - U_2$ solves the following Laplace equation

$$\begin{cases} \Delta V = \Delta U_1 - \Delta U_2 = \psi - \psi = 0, & \text{in } \Omega \\ V|_{\partial\Omega} = U_1|_{\partial\Omega} - U_2|_{\partial\Omega} = f - f = 0. \end{cases}$$

Then the maximum principle tells that $V \leq 0$ and $V \leq 0$. So $V \equiv 0$.

Namely $U_1 = U_2$ and the solution to the Poisson's equation is unique. □

2. Invariant properties of harmonic functions

There are some invariant properties of the solutions to the Laplace equations in either polar or Cartesian coordinates.

In Cartesian coordinates, we have

Proposition 2.1. *If $U(x, y)$ is a harmonic function on a disk of radius r centerred at the origin, then $V(x, y) = U(\lambda x, \lambda y)$ and $W(x, y) = U(\lambda x, -\lambda y)$ are both harmonic functions on a disk of radius $\frac{r}{\lambda}$.*

In polar coordinates, we have

Proposition 2.2. *If $U(r, \theta)$ is a harmonic function on $\mathbb{R}^2 \setminus \{0\}$, then both $V(r, \theta) = U(\frac{1}{r}, \theta)$ and $W(r, \theta) = U(\frac{1}{r}, -\theta)$ are harmonic functions on $\mathbb{R}^2 \setminus \{0\}$.*

Both of these 2 properties are left as exercises in the problem sets/Courseworks.

3. Basic concepts for the heat equation

In this chapter we study the $1+1$-dimensional heat equation —this is the paradigmatic example of parabolic equations:

$$U_t - \varkappa U_{xx} = 0,$$

with $\varkappa$ the so-called **diffusivity constant**. In $3+1$ dimensions the equation is given by

$$U_t - \varkappa\big(U_{xx} + U_{yy} + U_{zz}\big) = 0.$$

Thus, **time independent** solutions (i.e. with $U_t = 0$) satisfy the Laplace equation

$$\Delta U = 0.$$

We will interested in the following:

(i) **The boundary value problem**. Here one prescribes U at $t = 0$ and on $x = a$, $x = b$.
(ii) **The heat equation on the whole line.** In this case there are no boundary conditions and one only prescribes U at $t = 0$.

The heat equation has a wide range of applications in the study of heat propagation, diffusion of substances in a medium, finance, geometry...

3.1. **General remarks.** Consider the $1 + 1$ heat equation in the form

$$U_t = \varkappa U_{xx}, \qquad \varkappa > 0. \tag{3.1}$$

Geometrically, given a function $U(x,t)$, the second derivative U_{xx} is the rate of change of slope (at fixed time) —that is, it determines whether the graph of U (for fixed t) is concave or convex. On the other hand, U_t is the rate of change of $U(x,t)$ at some fixed point. Thus, one has that

$$\begin{aligned} U_t > 0 &\quad \text{if the graph of U(x,t) (for fixed } t\text{) is convex,} \\ U_t = 0 &\quad \text{if the graph is a straight line,} \\ U_t < 0 &\quad \text{if the graph is concave.} \end{aligned}$$

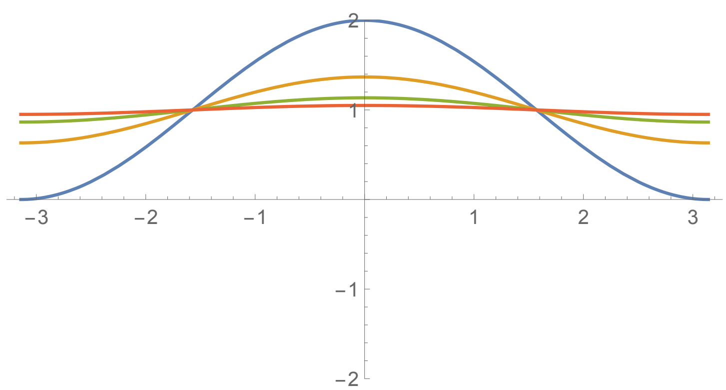
Thus, at all points x where $U_{xx} > 0$ we have that $U(x,t)$ increases in time , and at points where $U_{xx} < 0$ we have that $U(x,t)$ is decreasing in time.

Note. *The previous discussion shows that the effect of the heat equation is to* ***smooth out*** *bumps.*

Example 3.1. Consider the function

$$U(x,t) = 1 + e^{-\varkappa t} \cos x.$$

It can be checked to satisfy the heat equation. Plots of this function for various times are given below.



The plots show an initial central concentration spreading out an becoming more and more uniform as t increases. Observe, in particular how U increases where $U_{xx} > 0$ and decreases where $U_{xx} < 0$. Changing the value of $\varkappa$ affects the rate of smoothing: larger $\varkappa$ means faster smoothing and viceversa.

3.2. **Boundary conditions.** Recall that when solving first order ode's one needs one condition on the unknown (initial condition) to determine fully the solution. Since we want to predict the distribution of concentration/temperature $U(x,t)$ for all $t > 0$ and the heat equation has only one derivative in time, then at every x we need to prescribe one initial condition for $U(x,t)$ at $t = 0$ —that is

$$U(x,0) = f(x).$$

On the other hand, since $U_t = \varkappa U_{xx}$ contains U_{xx} and $x \in (a,b)$, we need to prescribe boundary conditions at the end points a and b at each time. This is consistent with the general principle for ode's that to solve second order boundary value problems one needs two boundary conditions (one at each point). These boundary conditions are determined by physical modelling and might contain U and U_x. The most common types are:

(i) **Dirichlet boundary conditions.** Here one prescribes

$$U(a,t) = h(t), \qquad U(b,t) = g(t).$$

These boundary conditions correspond to the temperature/concentration at the endpoints.

(ii) **Neumann boundary conditions.** Here one prescribes

$$U_x(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

In this case one prescribes a flux of U rather than U itself. In particular, if

$$U_x(a,t) = U_x(b,t) = 0,$$

the endpoints are insulated —i.e. no flux.

(iii) **Mixed boundary conditions.** One can also have situations as

$$U_x(a,t) = h(t), \qquad U(b,t) = g(t),$$

or

$$U(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

(iv) **Periodic boundary conditions.** One can also have

$$U(-a,t) = U(a,t)$$

or

$$U_x(-a,t) = U_x(a,t).$$

4. The heat equation on an interval

In this section we will see how the method of separation of variables can be used to obtain solutions to the heat equation on an interval. More precisely, we consider the following problem:

$$\begin{aligned}
&U_t = \varkappa U_{xx}, \qquad x \in [0,L], \quad t > 0,\\
&U(x,0) = f(x),\\
&U(0,t) = 0, \qquad U(L,t) = 0.
\end{aligned}$$

The boundary conditions describe, for example, a metallic wire whose ends are set (by means of some device) at a temperature of 0 degrees.

4.1. **Separation of variables.** Following the general strategy of the method we consider solutions of the form

$$U(x,t) = X(x)T(t).$$

Substitution into the heat equation gives

$$X(x)\dot{T}(t) = \varkappa X''(x)T(t).$$

Hence, dividing by XT we find that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)}.$$

We observe that the left hand side of this last expression only depends on x. The right hand side depends only on t. Thus, for the equality to hold one needs both sides to be constant. That is, one has that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

Thus, we end up with the following ordinary differential equations:

$$\dot{T} = -\varkappa - \lambda T, \tag{4.1a}$$

$$X'' = -\lambda X. \tag{4.1b}$$

Moreover, from the boundary conditions one has that

$$X(0)T(t) = 0, \qquad X(L)T(t) = 0,$$

so that

$$X(0) = X(L) = 0. \tag{4.2}$$

4.2. **Solving the equation for $X(x)$.** Combining equation (4.1b) with the boundary conditions (4.2) one obtains the eigenvalue problem

$$\begin{aligned} X'' &= -\lambda X, \\ X(0) &= X(L) = 0. \end{aligned}$$

Notice we have already proved the following claim about eigenvalues when studying wave equations.

Claim 4.1. *The eigenvalues* $\lambda \geq 0$.

Thus, the general solution to equation (4.1b) is given by

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Now, we make use of the boundary conditions. First we observe that

$$X(0) = A\cos 0 + B\sin 0.$$

Thus, from (4.2) it follows that

$$A = 0.$$

Using now $X(L) = 0$ one finds that

$$B\sin\lambda L = 0.$$

Clearly one needs $B \neq 0$ to get a non-trivial solution. Thus

$$\sqrt{\lambda} = \frac{\pi n}{L}, \qquad n = 1, 2, \ldots$$

Hence, the solution to the eigenvalue problem is given (ignoring the constant B) by

$$X_n(x) = \sin\left(\frac{\pi n x}{L}\right), \qquad \lambda_n = \frac{\pi^2 n^2}{L^2}.$$

4.3. **Solving the equation for $T(t)$.** Now knowing $\lambda = \frac{\pi^2 n^2}{L^2}$, we can solve for T

$$T_n(t) = Ce^{-\lambda_n \varkappa t} = Ce^{-\frac{\pi^2 n^2}{L^2}\varkappa t}, \qquad C \quad \text{a constant.}$$

4.4. **General solution.** The calculations from the previous sections can be combined to obtain the family of solutions to the heat equation

$$U_n(x,t) = X_n(x)T_n(t) = e^{-\frac{\pi^2 n^2}{L^2}\varkappa t}\sin\left(\frac{\pi n x}{L}\right).$$

The general solution is then applied using the principle of superposition:

$$U(x,t) = \sum_{n=1}^{\infty} a_n U_n = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2}\varkappa t}\sin\left(\frac{\pi n x}{L}\right). \tag{4.3}$$

4.5. **Initial conditions.** The a_n constants that are fixed through the initial conditions.
Evaluating (4.3) at $t = 0$ one has

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right).$$

This is a Fourier sine series —we have already found these series a couple of times before. The coefficients a_n are then determined via the Fourier coefficients —thus,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx.$$

4.6. **Examples.** We now look at some concrete examples of the discussion in the previous paragraphs.

Example 4.2. Let the initial conditions be given by

$$f(x) = \sin\left(\frac{\pi x}{L}\right).$$

It follows then that

$$U(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right).$$

Comparing the two sides of the last equality, and given that the sine functions in the infinite series are independent of each other one finds that

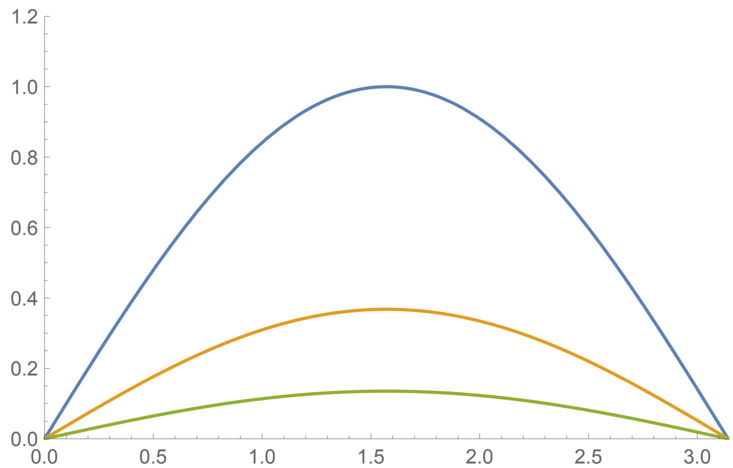
$$a_1 = 1, \qquad a_n = 0, \quad n \geq 2.$$

Thus, the particular solution to the heat equation is given by

$$U(x,t) = e^{-\pi^2 \varkappa t/L^2}\sin\left(\frac{\pi x}{L}\right).$$

A plot of the solution for various values of t is given below. Observe that

$$U(x,t) \to 0, \qquad \text{as} \quad t \to \infty.$$



Example 4.3. Let the initial conditions be given by

$$f(x) = 1, \qquad x \in [0, L].$$

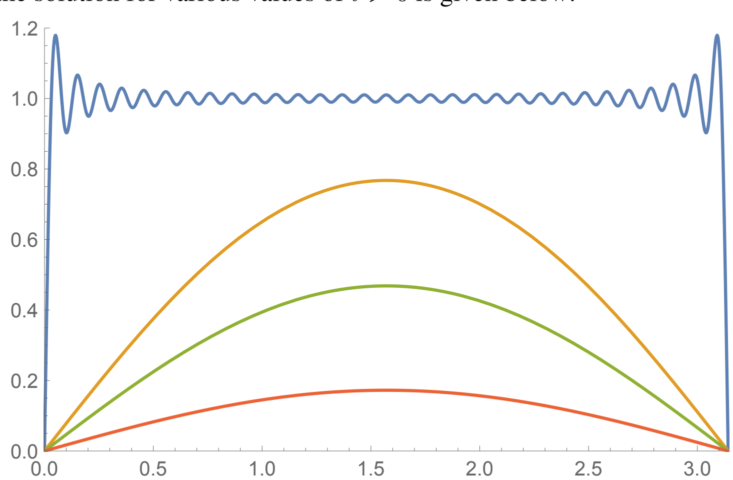
In this case we have to explicitly compute the Fourier coefficients —this is because the constant function does not appear in the series. One has that

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\ &= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

Hence, one can write

$$U(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} e^{-n^2\pi^2\varkappa t/L^2}\sin\left(\frac{n\pi x}{L}\right).$$

A plot of the solution for various values of $t > 0$ is given below:



Observe that the solution for $t > 0$ instantly drops to 0 at the ends. Observe that for $n \geq 3$ one has that

$$e^{-\pi^2 \varkappa t/L^2} >> e^{-9\pi^2 \varkappa t/L^2}.$$

Thus, one has that

$$U(x,t) \approx \frac{4}{\pi} e^{-\pi^2 \varkappa t/L^2} \sin\left(\frac{\pi x}{L}\right).$$

In other words, the first term in the infinite series dominates.

Example 4.4. Let $L = 1$ and

$$f(x) = \begin{cases} 1 & 0 < x < 1/2 \\ 0 & 1/2 < x < 1 \end{cases}.$$

Again, we need to compute explicitly the Fourier coefficients. In this case we have

$$a_n = 2\int_0^{1/2} \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos(\frac{n\pi}{2}) - 1\right).$$

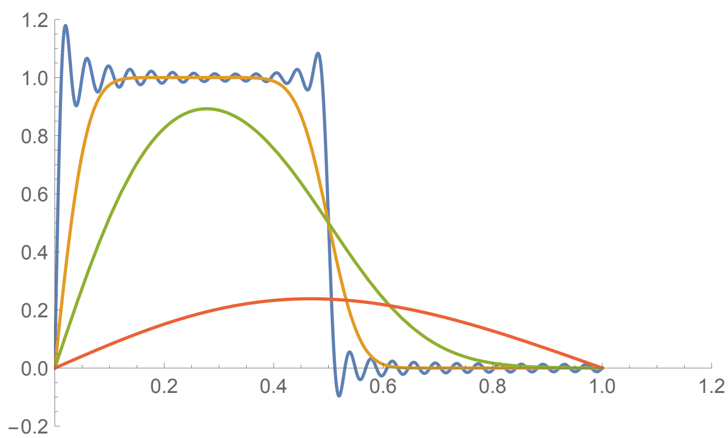
Hence,

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi/2}{n}\right) e^{-n^2\pi^2\varkappa t}\sin n\pi x.$$

Observe that

$$1 - \cos\frac{n\pi}{2} = 1,\ 2,\ 1,\ 0,\ 1,\ 2, \ldots.$$

A plot of the solution for various $t > 0$ is given below:



Observe that initially one has a step. The solution immediately becomes smooth. It gets more symmetric and sinusoidal as time increases.

Next, let's consider a mixed boundary condition problem for heat equations on an interval.

Example 4.5.

$$\begin{cases} U_t - U_{xx} = 0, x \in [0, \frac{\pi}{2}] \\ U(x,0) = 2\cos x \\ U_x(0,t) = 0, U(\frac{\pi}{2}, t) = 0. \end{cases}$$

Here we have the heat constant $\varkappa = 1$.

Suppose we have a separated variable solution $U(x,t) = X(x)T(t)$, we then get

$$X\dot{T} = X''T$$
$$\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda,$$

where λ is a constant. As before, this gives us 2 ODEs

$$X'' = -\lambda X$$
$$\dot{T} = -\lambda T.$$

Combing the ODE of X with the boundary conditions give us the following boundary value problem.

$$X'' = -\lambda X$$
$$X'(0) = 0, X(\frac{\pi}{2}) = 0.$$

As before, using the boundary conditions and integration by parts, we can show that the eigenvalues satisfy $\lambda > 0$ and so the general solutions for X is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$
$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

Using the first boundary condition, we get

$$0 = X'(0) = 0 + B\sqrt{\lambda}.$$

So $B = 0$ and $A \neq 0$. Using the second boundary condition, we get

$$0 = X'(L) = A\cos\sqrt{\lambda}L.$$

So $\sqrt{\lambda}L = n\pi - \frac{1}{2}\pi$.

We get the eigenvalues are $\lambda_n = (2n-1)^2$, for $n = 1, 2\ldots$ and the eigenfunctions are

$$X_n(x) = \cos[(2n-1)x].$$

Knowing λ_n, we can go back to solve the ODE $\dot{T} = -\lambda_n T$ for T and get

$$T_n(t) = e^{-(2n-1)^2 t}.$$

Then general solutions are

$$U(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2 t}\cos[(2n-1)x].$$

Next, we use the initial condition to specify the $a_n' s$. When $t = 0$, we have

$$2\cos x = U(x,0) = \sum_{n=1}^{\infty} a_n \cos[(2n-1)x].$$

By the orthogonality of the $\cos[(2n-1)x]$ trigonometric functions, we "observe" that $a_n = 0$ except for $n = 1$. Moreover, the $n = 1$ term have to match and so $a_1 = 2$. Thus the solution to this question is

$$U(x,t) = 2e^{-t}\cos x.$$

Example 4.6.

$$\begin{cases} U_t - U_{xx} = 0, x \in [0, \pi] \\ U(x, 0) = 1 + 2\cos x \\ U_x(0, t) = 0, U_x(\pi, t) = 0. \end{cases}$$

First look for solutions of the form $U = XT$ and get $X\dot{T} = X''T$, and so

$$\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda.$$

The eigenvalue problem for X is

$$\begin{cases} X'' = -\lambda X \\ X'(0) = X'(\pi) = 0. \end{cases}$$

We know that the eignevlaues are $\lambda \geq 0$.

If $\lambda > 0$, we have

$$\begin{cases} X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \\ X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x \end{cases}$$

$X'(0) = 0$ tells us that $B = 0$.

$X'(\pi) = 0$ tells us that $\sqrt{\lambda}\pi = n\pi$. So

$$\begin{cases} \lambda_n = n^2 \\ X_n(x) = \cos nx \end{cases}$$

for $n \geq 1$.

Knowing λ_n, we can solve for $\dot{T} = -\lambda T$ can get

$$T_n(t) = e^{-n^2 t}.$$

If $\lambda = 0$, we have $X(x) = ax + b$, the boundary conditions then gives $a = 0$. So

$$\begin{cases} \lambda_0 = 0 \\ X_0(x) = 1 \end{cases}$$

The corresponding solutions for T is $T_0(t) = 0$.

The general solutions are then

$$U(x, y) = \sum_{n=0}^{\infty} a_n X_n(x) T_n(t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos nt.$$

Using the initial conditions we have when $t = 0$ that

$$1 + 2\cos x = U(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt.$$

So we observe that $a_n = 0$ except for $n = 0, 1$. And $a_0 = 1, a_1 = 2$. The solution is then

$$U(x, t) = 1 + 2^{-t}\cos x.$$

5. Energy for heat equation on the interval and applications

Recall that the wave equation has an energy quantity that was preserved along time, which was useful in proving uniqueness of solutions to the wave equations. We can also define an energy quantity for the heat equation.

Consider the heat equation on the interval with fixed boundary condition from the last section

$$\begin{aligned}
&U_t = \varkappa U_{xx}, \qquad x \in [0, L], \quad t > 0,\\
&U(x, 0) = f(x),\\
&U(0, t) = 0, \qquad U(L, t) = 0.
\end{aligned}$$

We define the energy to be

$$E[U](t) = \frac{1}{2}\int_0^L U^2(x, t)dx.$$

Proposition 5.1. *The energy is non-increasing along time. It's preserved along time if and only if U is constant.*

Proof.

$$\begin{aligned}
\frac{d}{dt}E[U](t) =& \frac{d}{dt}\left[\frac{1}{2}\int_0^L U^2(x, t)dx\right]\\
=& \frac{1}{2}\int_0^L 2U(x, t)U_t(x, t))dx\\
=& \varkappa \int_0^L U(x, t)U_{xx}(x, t))dx\\
=& \varkappa U \cdot U_x|_0^L - \int_0^L \varkappa(U_x)^2dx\\
=& -\varkappa \int_0^L (U_x)^2dx\\
\leq& 0.
\end{aligned}$$

So the energy is non-increasing along time.

Since $U_x^2 \geq 0$, so the energy is preserved if and only if $U_x \equiv 0$.

If $U_x \equiv 0$, we have $U_{xx} \equiv 0$ and $U_t = \varkappa U_{xx} = \equiv 0$. So if the energy is preserved along time, U must be constant. □

By the Wirtinger inequality (we will use it here without proof): if f satisfies $f(0) = f(L) = 0$ on the interval $[0, L]$, we must have

$$\int_0^L [f^2(x)]dx \leq C_0 \int_0^L [f'(x)]^2dx,$$

for some universal constant $C_0 > 0$. Using this we have the energy satisfies

$$\begin{aligned}\frac{d}{dt}E[U](t) \leq& -\varkappa\int_0^L (U_x)^2dx\\ \leq& \frac{-\varkappa}{C_0}\int_0^L U^2dx\\ =& \frac{-2\varkappa}{C_0}E[U](t).\end{aligned}$$

This gives the decay of energy

$$E[U](t) \leq E[U](0)\cdot e^{\frac{-2\varkappa}{C_0}t} \to 0,$$

as $t \to \infty$.

As an application of the non-increasing of energy, we show the uniqueness of solutions of heat equations on interval with initial and boundary conditions.

Theorem 5.2. *Let U_1 and U_2 are 2 solutions to the following heat equation an interval with initial and Dirichlet boundary conditions:*

$$\begin{aligned}&U_t = \varkappa U_{xx}, x \in [0, L]\\ &U(x, 0) = f(x)\\ &U(0, t) = h(t), U(L, t) = g(t).\end{aligned}$$

Then we must have $U_1 \equiv U_2$.

Proof. Let $V(x,t) = U_1(x,t) - U_2(x,t)$. Then by the principle of superposition, we have V satisfies the equation

$$\begin{aligned}&V_t = \varkappa V_{xx}, x \in [0, L]\\ &V(x, 0) = 0\\ &V(0, t) = 0, V(L, t) = 0.\end{aligned}$$

The energy at time $t = 0$ is $E[V](0) = 0$. By the non-increasing of energy Proposition 5.1 $\frac{d}{dt}E[V](t) \leq 0$ and the fact that $E[V](t) \geq 0$, we must then have

$$E[V](t) \equiv 0.$$

So $V \equiv 0$ for all $t \geq 0$ and $U_1 = U_2$. □