Selected solutions for problem set 7.

1. First separate variables. Look for solutions of the form

$$u(x,y) = X(x)\,Y(y)$$

Get
$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

Get eigenvalue problems

$$(*1)\quad \begin{cases} Y'' + kY = 0 \\ Y(0) = 0 \\ Y(b) = 0 \end{cases}$$

and
$$(*2)\quad \begin{cases} X'' - kX = 0. \\ X(0) = g_1(y) \\ \cancel{X(b) = 0}\ X(a) = 0 \end{cases}$$

claim: $k \geq 0$

proof of claim:
$$\int_0^b Y \cdot Y'' + kY^2 = 0$$
Multiply equation by X and integrate

$$YY'\Big|_0^b - \int_0^b (Y')^2 + k\int_0^b Y^2 = 0$$

~~so $k \geq 0$.~~
$$k\int_0^b Y^2 = \int_0^b (Y')^2$$

So $k \geq 0$.

Thus the general solutions are

$$Y(y) = A\cos(\sqrt{k}\,y) + B\sin(\sqrt{k}\,y)$$

$$\cancel{= A\cos(my) + B\sin(my),}$$ ~~where write $k = m^2 \geq 0$.~~

using $Y(0)=0$, we get

$$A = Y(0) = 0$$

so $B \neq 0$

using $Y(b)=0$, we get

$$B\sin(\sqrt{k}\,b) = Y(b) = 0$$

Thus $\sqrt{k}\,b = n\pi$, $\quad n = 1, 2, \cdots$

Next solve $X(x)$, get

$$X'' - \frac{n^2\pi^2}{b^2} X = 0.$$

So $\quad X_n(x) = \widetilde{C_n}\cosh\left(\frac{n\pi(x-a)}{b}\right) + \widetilde{D_n}\sinh\left(\frac{n\pi(x-a)}{b}\right)$

using that ~~$X(b)=0$,~~ $X(a)=0$

we get $\quad \widetilde{C_n} = 0$ and

$$X_n(x) = \widetilde{D_n}\sinh\left(\frac{n\pi(x-a)}{b}\right)$$

So $U_n(x,y) = X_n(x)\,Y_n(y) = \widetilde{D_n}\,B_n \sin\frac{n\pi y}{b}\sinh\left[\frac{n\pi(x-a)}{b}\right]$

and the general solutions are

$$U(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi(x-a)}{b}$$

where $\quad D_n = \widetilde{D_n}\,B_n$

To fix D_n, we use $U(0,y) = g_1(y)$, i.e.

$$\sum_{n=1}^{\infty} -D_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi a}{b} = g_1(y)$$

So $\quad D_n = -\dfrac{\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy}{\frac{b}{2}\sinh\frac{n\pi a}{b}}$

And the solution to this Dirichlet problem is

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy}{\frac{b}{2}\sinh\frac{n\pi a}{b}} \sin\frac{n\pi y}{b}\sinh\frac{n\pi(x-a)}{b}.$$

3. we use the principle of superposition and consider the following 2 problems.

$$(*_3)\begin{cases} \Delta v = 0 \\ v(x,0) = x \\ v(x,1) = 0 \\ v(0,y) = 0 \\ v_x(1,y) = 0 \end{cases} \qquad (*_4)\begin{cases} \Delta w = 0 \\ w(x,0) = 0 \\ w(x,1) = 0 \\ w(0,y) = 0 \\ w_x(1,y) = y^2 \end{cases}$$

First. solve $(*_3)$ by separation of variables.

suppose. $v(x,y) = \cancel{v(x)v(y)}$ $X(x)Y(y)$

get $\quad \frac{X''}{X} = -\frac{Y''}{Y} = k$

and $\begin{cases} X'' - kX = 0 \\ X(0) = 0 \\ X'(1) = 0 \end{cases} \qquad \begin{cases} Y'' + kY = 0 \\ Y(0) = x \\ Y(1) = 0. \end{cases}$

claim: $k \leq 0$.

proof of claim: $\quad \int_0^1 XX'' - \int_0^1 kX^2 = 0$ Multiply equation by X and integrate.

$$XX'\Big|_0^1 - \int_0^1 (X')^2 - k\int X^2 = 0$$

$=0$ by boundary condition. $\quad k\int_0^1 X^2 = -\int (X')^2$

so $k \leq 0$

So $X(x) = A\cos(\sqrt{-k}\,x) + B\sin(\sqrt{-k}\,x)$.

using $X(0)=0$, we get $A=0$

and so $B\neq 0$.

using $X'(1)=0$, we get

$$X'(1) = B\sqrt{-K}\cos(\sqrt{-K}x)\Big|_{x=1} = 0.$$

$$\sqrt{-K} = \frac{\pi}{2}+n\pi \quad \text{for } n=1,2,\ldots$$

$$K = -\left(\frac{1}{2}+n\right)^2\pi^2$$

Next solve $Y(y)$, get

$$Y'' - \left(\frac{1}{2}+n\right)^2\pi^2 Y = 0$$

~~$Y(y) = \tilde{C}_n \cos\left[\left(\frac{\pi}{2}+n\pi\right)y\right] + \tilde{D}_n \sin$~~

$$Y(y) = \tilde{C}_n \cosh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right] + \tilde{D}_n \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

Using that $Y(1)=0$, get

$$\tilde{C}_n = 0,$$

So $V_n(x,y) = X_n(x)\,Y_n(y)$

$$= B_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]\cdot \tilde{D}_n \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

$$= D_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

where $D_n = B_n\cdot\tilde{D}_n$

and the General solutions are

$$V(x,y) = \sum_{n=1}^{\infty} D_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]\sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

To fix D_n, we use $V(x,0)=x$, i.e.

$$x = \sum_{n=1}^{\infty} -D_n \sin\left(\left(\frac{\pi}{2}+n\pi\right)x\right)\cdot\sinh\left(\frac{\pi}{2}+n\pi\right)$$

so $$D_n = -\frac{\int_0^1 x \cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] dx}{\frac{1}{2} \cdot \sin\left(\frac{\pi}{2}+n\pi\right)}$$

$$= -2 \int_0^1 x \cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] dx$$

and the solution is

$$V(x,y) = -\sum_{n=1}^{\infty} \left\{ 2\int_0^1 x \cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] dx \right\} \sin\left(\frac{\pi}{2}+n\pi\right)x \cdot \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

~~Step~~ After getting V, we solve (*4) to get W.

similarly, suppose $W(x,y) = X(x)\,Y(y)$

get $$\frac{X''}{X} = -\frac{Y''}{Y} = k.$$

and $\begin{cases} X'' - kX = 0 \\ X(0) = 0 \\ X'(1) = y^2 \end{cases}$, $\begin{cases} Y'' + kY = 0 \\ Y(0) = 0 \\ Y(1) = 0. \end{cases}$

First solve Y,

$\boxed{\text{claim: } k \geq 0. \quad (\text{The proof is the same as in lecture})}$

get $Y(y) = A\cos(\sqrt{k}\,y) + B\sin(\sqrt{k}\,y)$

using $Y(0) = 0$, get $A = 0$, $B \neq 0$

using $Y(1) = 0$, get $\sqrt{k} = n\pi$,

so $k = n^2\pi^2$, $n = 0, 1, \cdots$

Next solve X,

$$X'' - n^2\pi^2 X = 0$$

so $X_n(x) = \tilde{C_n}\cosh(n\pi x) + \tilde{D_n}\sinh(n\pi x)$

Using that $X(0)=0$ get $\tilde{C_n}=0$. ⑥

So ~~solution for W is~~ general solution for W is

$$W(x,y) = \sum_{n=0}^{\infty} X_n(x)\, Y_n(y)$$

$$= \sum_{n=0}^{\infty} D_n \sin(n\pi y)\sinh(n\pi x)$$

where $D_n = B_n \cdot \overline{D_n}$.

To fix D_n, we use

$$y^2 = W_x(1,y) = \sum_{n=0}^{\infty} \cancel{D_n \cdot (n\pi)\cdot \cos(n\pi y)}\ \mathrm{si}$$
$$D_n\cdot(n\pi)\sin(n\pi y)\cdot\cosh(n\pi x)\Big|_{x=1}.$$

namely

$$y^2 = \sum_{n=0}^{\infty} D_n\cdot(n\pi)\cdot\frac{e^{n\pi}+e^{-n\pi}}{2}\cdot\sin(n\pi y)$$

thus
$$D_n = \frac{\int_0^1 y^2\cdot\sin(n\pi y)\,dy}{\frac{1}{2}\cdot(n\pi)\cdot\frac{e^{n\pi}+e^{-n\pi}}{2}}$$

and
$$W(x,y) = \sum_{n=0}^{\infty}\frac{4}{(n\pi)\cdot(e^{n\pi}+e^{-n\pi})}\cdot\left[\int y^2\sin(n\pi y)\,dy\right]\sin(n\pi y)\sinh(n\pi x)$$

The solution to the original problem is

$$u(x,y) = v(x,y)+W(x,y)$$
$$= -\sum_{n=1}^{\infty} 2\cdot\left\{\int_0^1 x\cdot\sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]dx\right\}\cdot\sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]\sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$
$$+\sum_{n=0}^{\infty}\frac{4}{(n\pi)(e^{n\pi}+e^{-n\pi})}\cdot\left[\int y^2\sin(n\pi y)\,dy\right]\sin(n\pi y)\sinh(n\pi x).$$

5. $$u(r,\theta) = \frac{(r^2 - r_*^2)}{2\pi} \int_0^{2\pi} \frac{f(\theta')\,d\theta'}{r^2 - 2rr_*\cos(\theta-\theta') + r_*^2}$$

$$= \frac{(r^2 - r_*^2)u_1}{2\pi} \int_0^{\pi} \frac{d\theta'}{r^2 + r_*^2 - 2rr_*\cos(\theta-\theta')}$$

$$+ \frac{(r^2 - r_*^2)u_2}{2\pi} \int_\pi^{2\pi} \frac{d\theta'}{r^2 + r_*^2 - 2rr_*\cos(\theta-\theta')}$$

Using the hint, we get

$$u(r,\theta) = -\frac{(r^2 - r_*^2)u_1}{2\pi} \cdot \frac{2}{r_*^2 - r^2} \arctan\left[\frac{r_*+r}{r_*-r} \tan\left(\frac{\theta-\theta'}{2}\right)\right]\Bigg|_0^{\pi}$$

$$- \frac{(r^2 - r_*^2)u_2}{2\pi} \cdot \frac{2}{r_*^2 - r^2} \arctan\left(\frac{r_*+r}{r_*-r} \tan\left(\frac{\theta-\theta'}{2}\right)\right)\Bigg|_\pi^{2\pi}$$

$$= -\frac{u_1}{\pi} \arctan\left(\frac{r_*+r}{r_*-r} \cot\frac{\theta}{2}\right) + \frac{u_1}{\pi} \arctan\left(\frac{r_*+r}{r_*-r} \tan\frac{\theta}{2}\right)$$

$$+ \frac{u_2}{\pi} \arctan\left(\frac{r_*+r}{r_*-r} \tan\frac{\theta}{2}\right) + \frac{u_2}{\pi} \arctan\left(\frac{r_*+r}{r_*-r} \cot\frac{\theta}{2}\right)$$

6. using the general formula in polar coordinates

$$u(r,\theta) = (C_0 + D_0 \ln r) + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(\cos m\theta + \sin m\theta).$$

since $u(r_1,\theta) = u_1$, $u(r_2,\theta) = u_2$ are independent of θ

we have all $C_m, D_m = 0$ for $m \geq 1$.

and

$$C_0 + D_0 \ln r_1 = u_1$$

$$C_0 + D_0 \ln r_2 = u_2.$$

$$\Rightarrow \quad C_0 = \frac{\ln\left(\frac{r_2^{u_1}}{r_1^{u_2}}\right)}{\ln\left(\frac{r_1}{r_2}\right)}$$

$$D_0 = \frac{u_2 - u_1}{\ln\left(\frac{r_1}{r_2}\right)}$$

$$u(r,\theta) = \frac{\ln\left[\frac{r_2^{u_1}}{r_1^{u_2}}\right]}{\ln\left(\frac{r_1}{r_2}\right)} + \frac{u_2 - u_1}{\ln\left(\frac{r_1}{r_2}\right)} \ln r.$$

PS 7 Q 4 :

The general solutions to Laplace equation in a disc of radius r_* is (as deduced in Week 8 Notes)

$$u(x,t) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

with bondry condition

$$f(\theta) = u(r_*, \theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta)$$

Here $f(\theta) = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

so using the fact that $\cos(m\theta)$ and $\sin n\theta$ are independent, we observe that

$$a_0 = \frac{1}{2}, \quad r_*^2 \cdot a_2 = -\frac{1}{2} \Rightarrow a_2 = -\frac{1}{2r_*^2}$$

all other a_n and b_n are zero.

so

$$u(x,t) = \frac{1}{2} - \frac{r^2}{2r_*^2}\cos(2\theta)$$

PS7 Q7 : (10 points)

Method 1. By the expression for general solutions of Laplace equations in polar coordinates, u must be of the form

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty} \left(C_m \cdot r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

and thus $u(\frac{1}{r}, \theta)$ must be of the form

(by replace r with $\frac{1}{r}$ above)

$$u(\tfrac{1}{r},\theta) = C_0 + D_0 \ln(r^{-1}) + \sum_{m=1}^{\infty}\left(\frac{C_m}{r^m} + D_m r^m\right)(A_m \cos m\theta + B_m \sin m\theta)$$

so if we choose

$$\widehat{C_0} = C_0,\ \widehat{D_0} = -D_0,\ \widehat{C_m} = D_m,\ \widehat{D_m} = C_m,\ \widehat{A_m} = A_m,\ \widehat{B_m} = B_m,$$

we will have

$$u(\tfrac{1}{r},\theta) = \widehat{C_0} + \widehat{D_0}\ln r + \sum_{m=1}^{\infty}\left(\widehat{C_m} r^m + \frac{\widehat{D_m}}{r^m}\right)(\widehat{A_m}\cos m\theta + \widehat{B_m}\sin m\theta)$$

is also of the form of a solution to Laplace eqution, so $u(\frac{1}{r},\theta)$ is also harmonic.

Method 2.

$$\Delta\left[u\left(\tfrac{1}{r},\theta\right)\right]$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[u\left(\tfrac{1}{r},\theta\right)\right]$$

$$= \frac{\partial^2}{\partial r^2}\left[u\left(\tfrac{1}{r},\theta\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[u\left(\tfrac{1}{r},\theta\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[u\left(\tfrac{1}{r},\theta\right)\right]$$

$$= \frac{\partial}{\partial r}\left[u_r\left(\tfrac{1}{r},\theta\right)\cdot\frac{-1}{r^2}\right] + \frac{1}{r}u_r\left(\tfrac{1}{r},\theta\right)\frac{-1}{r^2} + \frac{1}{r^2}u_{\theta\theta}\left(\tfrac{1}{r},\theta\right)$$

$$= u_{rr}\left(\tfrac{1}{r},\theta\right)\cdot\left(\frac{-1}{r^2}\right)^2 + \frac{2}{r^3}u_r\left(\tfrac{1}{r},\theta\right) - \frac{1}{r^3}u_r\left(\tfrac{1}{r},\theta\right) + \frac{1}{r^2}u_{\theta\theta}\left(\tfrac{1}{r},\theta\right)$$

$$= \frac{1}{r^4}u_{rr}\left(\tfrac{1}{r},\theta\right) + \frac{1}{r^3}u_r\left(\tfrac{1}{r},\theta\right) + \frac{1}{r^2}u_{\theta\theta}\left(\tfrac{1}{r},\theta\right)$$

$$= \frac{1}{r^4}\left[u_{rr}\left(\tfrac{1}{r},\theta\right) + r\,u_r\left(\tfrac{1}{r},\theta\right) + r^2 u_{\theta\theta}\left(\tfrac{1}{r},\theta\right)\right]$$

$$= \frac{1}{r^4}\left[u_{rr}\left(\tfrac{1}{r},\theta\right) + \frac{1}{\frac{1}{r}}u_r\left(\tfrac{1}{r},\theta\right) + \frac{1}{\left(\frac{1}{r}\right)^2}u_{\theta\theta}\left(\tfrac{1}{r},\theta\right)\right]$$

choosing $(r,\theta) = \left(\frac{1}{r_0},\theta_0\right)$, we get

$$\Delta\left[u\left(\tfrac{1}{r},\theta\right)\right]$$

$$= r_0^4\left[u_{rr}(r_0,\theta_0) + \frac{1}{r_0}u_r(r_0,\theta_0) + \frac{1}{r_0^2}u_{\theta\theta}(r_0,\theta_0)\right]$$

$$= r_0^4 \cdot 0$$

$$= 0$$

The bracket is zero because $u(r,\theta)$ is harmonic at (r_0,θ_0)

Since (r_0,θ_0) can be chosen arbitrary, we see $u\left(\frac{1}{r},\theta\right)$ is also harmonic!

PS 7 Q8:

The general solutions of Laplace equation in polar coordinate is

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

plug in the 2 boundary conditions, we get by "observation" that

the constant term has to agree, i.e.

$$\begin{cases} 1 = C_0 + D_0 \cdot \ln 1 = C_0 \\ 3 = C_0 + D_0 \cdot \ln e = C_0 + D_0 \end{cases}$$

so $\Rightarrow$ $C_0 = 1$, $D_0 = 2$

Next, because there are only $\sin\theta$ terms on the boundary data.

Using the fact that $\sin(m\theta)$ $\cos(n\theta)$ are independent, we must have.

$A_m = 0$ for all m

$B_m = 0$ for all m except for $m = 1$

Moreover, the coefficients before $\sin\theta$ need to match, so

$$\begin{cases} e = (C_1 \cdot 1 + D_1 \cdot 1)\, B_1 \\ 1 = (C_1 e + \frac{D_1}{e})\, B_1 \end{cases}$$

This equation has more unknowns than restrictions, we can set $B_1 = 1$, and get

$$\begin{cases} e = C_1 + D_1 \\ 1 = e \cdot C_1 + \frac{1}{e} \cdot D_1 \end{cases}$$

Solve it, we get $\begin{cases} C_1 = 0 \\ D_1 = e \end{cases}$

So the solution is

$$u(r, \theta) = 1 + 2\ln r + \frac{e}{r}\sin\theta$$