Queen Mary
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Mth6106: Group Theory (Mid-term Solutions)

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Question 1 [10 marks].

(a) **Solution**: An Abelian group G is a group for which the elements commute i.e., $ab = ba$ for all elements $a, b \in G$.

All cyclic groups are Abelian, but an Abelian group is not necessarily cyclic. Set of integers $(\mathbb{Z}, +)$ under addition is an example of abelian groups. The set of real numbers $(\mathbb{R}, +)$ under addition is also an abelian group.

In an Abelian group, each element is in a conjugacy class of itself. [**2**]

(b) **Solution**: We ll have

$$\begin{pmatrix} x_1 & y_1 \\ 0 & 1/x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ 0 & 1/x_2 \end{pmatrix} = \begin{pmatrix} x_3 & y_3 \\ 0 & 1/x_3 \end{pmatrix}$$

where $x_3 = x_1 x_2$ and $y_3 = x_1 y_2 + y_1/x_2$. Hence M is closed under multiplication. Matricxmultiplication is associative (as given). The identity element corresponds to $x = 1, y = 0$. For the inverse

$$\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y \\ 0 & 1/x \end{pmatrix} \in M$$

Its non abelian, since for example

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1/2 \end{pmatrix} \neq \begin{pmatrix} 2 & 1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

[**2**]

(c) **Solution**: H is non empty since $e^n = e$ is in H. Suppose $x, y \in H$, then $x^n = e$ and $y^n = e$.

$$\begin{aligned} (xy^{-1})^n &= (xy^{-1})(xy^{-1}) \cdots (xy^{-1}) && \text{n-times} \\ (xy^{-1})^n &= (x^n)(y^{-1})^n && \text{since G is abelian} \\ (xy^{-1})^n &= (x^n)(y^n)^{-1} = e.e^{-1} = e.e = e \end{aligned}$$

If G is non-abelian then $(xy)^n = (xy)(xy)(xy) \cdots (xy)$, H cannot be closed. [**3**]

(d) **Solution**: The only possibility for ? in row 2 is c, otherwise there would be two c's in the last column.

$$\begin{pmatrix} e & a & b & c \\ a & e & ? & ? \\ b & ? & e & ? \\ c & ? & ? & ? \end{pmatrix}$$

Hence

$$\begin{pmatrix} e & a & b & c \\ a & e & c & b \\ b & ? & e & ? \\ c & ? & ? & ? \end{pmatrix}$$

Again ? in row 3 must be c, and thus

$$\begin{pmatrix} e & a & b & c \\ a & e & c & b \\ b & c & e & a \\ c & b & a & e \end{pmatrix}$$

Hence the group table is completely determined by the relations $a^2 = b^2 = e$. The associativity of the composition law can easily be checked. The resulting group is called Klein four group. [3]

Question 2 [10 marks].

(a) **Solution**: **Cosets as equiavlence classes.** Suppose G is a group with a subgroup H. We define a relation $(x, y) \in R_H$ if and only if $xy^{-1} \in H$.

This is ean equivalence relation.

Reflexive: as $xx^{-1} = 1 \in H$ implies $(x, x) \in R_H$.

Symmetric: suppose $(x, y) \in R_H$ then $xy^{-1} \in H$, as H is a subgroup, it follows that $(xy^{-1})^{-1} = yx^{-1} \in H$ and thus $(y, x) \in H$.

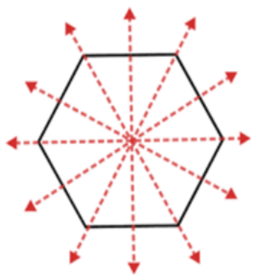
Transitive: if $xy^{-1} \in H$ and $yz^{-1} \in H$ then $xy^{-1}yz^{-1} \in H$ as H is closed as a subgroup, this means $(x, z) \in H$. [3]

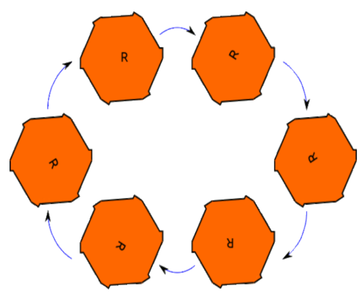
(b) The dihedral group of order 12, denoted as $\mathcal{D}_{12}$, is the group of symmetries of a hexagon, consisting of rotations r of order 6 and reflections s of order 2 considering as six axes of symmetry.

(i) **Solution**:

$$\mathcal{D}_{12} = \{1, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$$

[2]





(ii) **Solutions**: The set $\langle r^2 \rangle$ is generated by r^2 a rotation by 180°.
The powers of r^2 are $r^2.r^2 = r^4$, $\quad r^2.r^4 = e$ and $r^4.r^4 = r^2$ so $\langle r^2 \rangle$ is a cyclic subgroup of order 3.

$$\langle r^2 \rangle = \{e, r^2, r^4\}$$

The inverse of r^2 is r^4, since $r^2.r^4 = e$, similarly inverse of r^4 is r^2.

This set is closed under multiplication, contains identity element, and each element has an inverse. This is of order 3 which divides 12 (using Lagranges' tHeorem) This is a subgroup of $\mathcal{D}_{12}$. [**2**]

(iii) **Solutions**: The order of H is 3, therefore the index of H is 4, so there will be 4 cosets of H in $\mathcal{D}_{12}$.
the subgroup itself a coset: $eH = \{e, r^2, r^4\}$
left coset from r: $rH = \{r, r^3, r^5\}$
left coset from s: $sH = \{s, sr^2, sr^4\}$
left coset from sr: $srH = \{sr, sr^3, sr^5\}$ [**3**]

Question 3 [10 marks].

(a) Consider the two permutations in $\mathcal{S}_8$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 8 & 4 & 3 & 6 & 7 & 5 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 1 & 3 & 2 & 5 & 6 & 8 & 4 \end{pmatrix}.$$

Solution: We get $f = (128)(34)(567)$ and $g = (17842)$ which leads to $fg = (243)(5687)$ and $gf = (1567)(348)$. Using the method from Lemma 2.4 we can write

$$f = (12)(28)(34)(56)(67)$$

$$g = (17)(78)(84)(42)$$

so that f is odd and g is even. It follows that fg and gf must be odd, so only g belongs to $\mathcal{A}_8$ and the others do not.

Order of f is 6, order of g is 5 and order of fg is 12. [**4**]

(b) Let G be a group, let $f, g \in G$, and suppose that $f \sim_G g$. Show that the order of f is equal to the order of g. **[3]**

Solution: Suppose first that f, g are conjugate. Then there is $k \in \mathcal{S}_n$ such that $g = kfk^{-1}$.

Suppose $(a_1 \ \dots \ a_r)$ is a cycle of f. This means

$$f \cdot a_1 = a_2, \qquad f \cdot a_2 = a_3, \qquad \dots \qquad f \cdot a_r = a_1$$

which gives

$$k^{-1}gk \cdot a_1 = a_2, \qquad k^{-1}gk \cdot a_2 = a_3, \qquad \dots \qquad k^{-1}gk \cdot a_r = a_1.$$

Applying k throughout, we get

$$g \cdot (k \cdot a_1) = k \cdot a_2, \quad g \cdot (k \cdot a_2) = k \cdot a_3, \qquad \dots \qquad g \cdot (k \cdot a_r) = k \cdot a_1.$$

And this means that $(k \cdot a_1 \ \dots \ k \cdot a_r)$ is a cycle of g. So for every cycle of f, there is a corresponding cycle of g of the same length. So the list of the lengths of the cycles of f is the same as the list of the lengths of the cycles for g, i.e. f and g have the same cycle type. Conversely, suppose f and g have the same cycle type. Then we can pair up the cycles of f with the cycles of g so that each cycle of f is paired with a cycle of g of the same length. Say we've paired up the cycle $(a_1 \ \dots \ a_r)$ of f with the cycle $(b_1 \ \dots \ b_r)$ of g; we define $k \cdot a_i = b_i$ for each i. Then for each i we have

$$kfk^{-1} \cdot b_i = kf \cdot a_i = k \cdot a_{i+1} = b_{i+1} = g \cdot b_i. \qquad (*)$$

Doing this for every cycle, we get a function $k : \{1, \dots, n\} \to \{1, \dots, n\}$ (because every number in $\{1, \dots, n\}$ appears in exactly one cycle of f). Furthermore, k is a bijection (because every number in $\{1, \dots, n\}$ appears in exactly one cycle of g); that is, $k \in \mathcal{S}_n$. By $(*)$ we have $kfk^{-1} \cdot b = g \cdot b$ for all $b \in \{1, \dots, n\}$, so $kfk^{-1} = g$. Since f and g have same cycle types, they would have same l.c.m and hence same order.

(c) We have

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 6 & 5 & 1 & 7 & 4 & 3 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 5 & 1 & 2 & 4 & 7 & 3 \end{pmatrix} \text{ in } \mathcal{S}_n$$

Since

$$f = (1264)(357)$$

and

$$g = (3167)(254)$$

Yes, f and g are conjugate as they have same cylce type. We ll have $k = (132)(47)$ **[3]**

Question 4 [10 marks].

(a) (i) **Solution**:

$$\begin{aligned} G/N \text{ is abelian} &\iff (Nf)(Ng) = (Ng)(Nf) \quad \text{for all } f,g \in G \\ &\iff Nfg = Ngf \quad \text{for all } f,g \in G \\ &\iff fgf^{-1}g^{-1} \in N \quad \text{for all } f,g \in G \qquad \text{by the Coset Lemma.} \end{aligned}$$

So G/N is abelian if and only if N contains every commutator. But N is closed under multiplication, so if it contains every commutator, then it contains G'. [**2**]

(ii) Give an example of a group G and two subgroups H_1, H_2 such that $H_1H_2 \neq H_2H_1$ and H_1H_2 is not a subgroup of G. [**3**]

Solution: Take $G = \mathcal{S}_3$, $H_1 = \{id, (12)\}$ and $H_2 = \{id, (23)\}$. Then

$$H_1H_2 = \{id, (1\,2), (2\,3), (1\,2\,3)\}.$$

$$H_2H_1 = \{id, (1\,2), (2\,3), (1\,3\,2)\}$$

(This is not a subgroup of $\mathcal{S}_3$, as we can see straight away from Lagrange's Theorem.)

(i) **Solution**: G is a group and $A \subseteq G$, then the **centraliser** of A in G.

$$C_G(A) = \{g \in G \,|\, \forall a \in A : gag^{-1} = a\}$$

[**2**]

(ii) **Solution**: All rotations commute with r^3 so the centraliser contains all five rotations (including the identity). Since centraliser of a subset of a group is a subgroup itself. It follows by Lagrange's theorem that the centraliser either consists only of rotations, or consists of all elements of $\mathcal{D}_{10}$.
Since $r^3s = sr^{-3} = sr^2 \neq sr^3$, r^3 does not commute with s, so the centraliser is not the whole of $\mathcal{D}_{10}$ and therefore must be $\{1, r, r^2, r^3, r^4\}$. [**3**]