Queen Mary
University of London

Mth6106: Group Theory (Mid-term)

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This coursework counts for 20% of your mark for this module. You should answer all questions, and each question will be marked out of 10. You should give full explanation of your answers. Please submit your solutions on QMPlus by 5pm on Friday 15th November 2024. Your submission must be entirely your own work.

Question 1 [10 marks].

(a) Give an example of an abelian group G. If g is an element of an abelian group G, write down the conjugacy class of g. **[2]**

(b) Given the set of matrices

$$M = \left\{ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} : \quad x \in \mathbb{R}^*, y \in \mathbb{R} \right\}$$

where the composition $\circ$ is matrix multiplication. Show that M is a group. (You can assume that matrix multiplication is associative.) Check if this group is abelian or non-abelian. **[2]**

(c) Let G be an abelian group and n a positive integer, and let

$$H = \left\{ x \in G : \; x^n = e \right\}$$

Prove that the set H is a subgroup of G.

Give an example to show that this conclusion may not be true if G is a non-abelian group. **[3]**

(d) Let $G = \left\{ e, a, b, c \right\}$ and assume $a^2 = b^2 = e$. Complete the following group table of G. **[3]**

$$\begin{pmatrix} e & a & b & c \\ a & e & ? & ? \\ b & ? & e & ? \\ c & ? & ? & ? \end{pmatrix}$$

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Question 2 [10 marks].

(a) Given a subgroup H of a group G we define the relation $R_H \subseteq G \times G$ by

$$(x, y) \in R_H \Leftrightarrow xy^{-1} \in H.$$

Show that the relation R_H defines an equivalence relation in G. **[3]**

(b) This question is about the dihedral group $\mathcal{D}_{12}$, the group of symmetries of a regular hexagon.

(i) Describe all the elements of $\mathcal{D}_{12}$ (draw a diagram like we did for $\mathcal{D}_8$ in lectures, and label the axes of reflections and mention the rotations in terms of r and s). **[2]**

(ii) Let $H = \langle r^2 \rangle$. List all the elements of H, give reason why it is a subgroup. **[2]**

(iii) Give a list of all **left** cosets of H in G. (Hint: cosets generated by s, etc). **[3]**

Question 3 [10 marks].

(a) Consider the two permutations in S_8 given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 8 & 4 & 3 & 6 & 7 & 5 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 1 & 3 & 2 & 5 & 6 & 8 & 4 \end{pmatrix}.$$

Write down f, g, fg and gf in disjoint cycle notation. Write these permutations as products of transpositions and decide which of these belong to $\mathcal{A}_8$ and which do not. **[4]**

(b) Let G be a group, let $f, g \in G$, and suppose that $f \sim_G g$. Show that the order of f is equal to the order of g. **[3]**

(c) Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 6 & 5 & 1 & 7 & 4 & 3 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 5 & 1 & 2 & 4 & 7 & 3 \end{pmatrix}.$$

are elements of $\mathcal{S}_7$. Check if f and g are conjugate in $\mathcal{S}_7$? Give reason. If they are conjugate find out the value of k. **[3]**

Question 4 [10 marks].

(a) (i) Let G be a group and N a subgroup of G. Prove that N is normal in G if and only if the set of right cosets coincide with the set of left cosets, i.e.$\forall g \in G \quad gN = Ng$ [**2**]

(ii) Give an example of a group G and two subgroups H_1, H_2 such that $H_1H_2 \neq H_2H_1$ and H_1H_2 is not a subgroup of G. [**3**]

(b) (i) Define what is meant by the **centraliser** of a subset A of a group G. [**2**]

(ii) Consider the element r^3 of the dihedral group $\mathcal{D}_{10}$. Find the centraliser of r^3 in $\mathcal{D}_{10}$. [**3**]