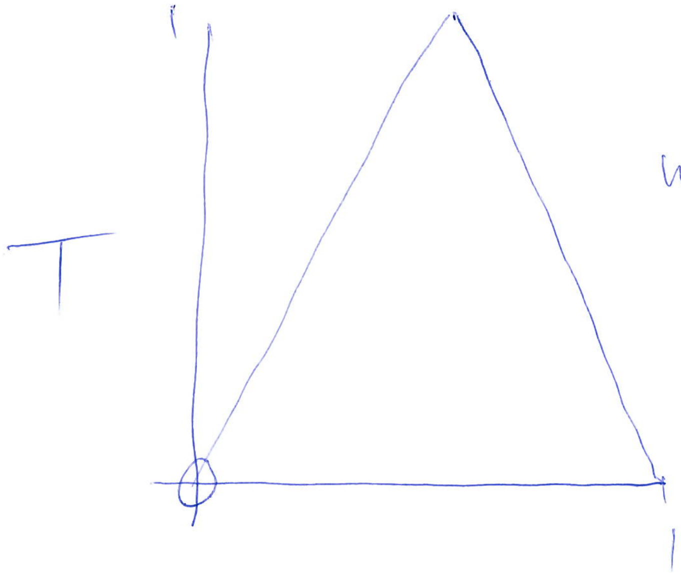
Let's look at periodic orbits of the tent map T (since T is piecewise linear, it is easier to work with iterates T^n, thus easier to (explicitly) find period-n points for T rather than for f_4).

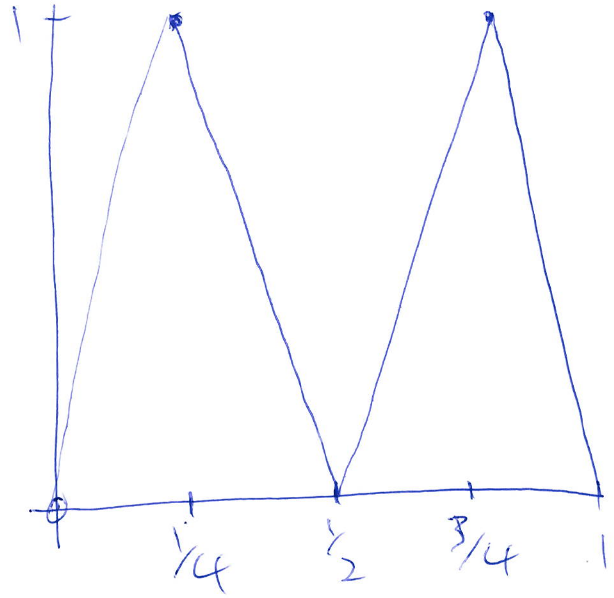
<u>Question</u> When does $T^n(x)$ equal 0 or 1? (Motivation: Since T^n is continuous, we will argue that if $T^n(a) = 0$ and $T^n(b) = 1$ then there is a fixed point of T^n in $[a, b]$, i.e. a period-n point for T).

$n = 1$



$T(x) = 0$
when $x = 0, 1$
<u>and</u>
$T(x) = 1$
when $x = \frac{1}{2}$

$n = 2$



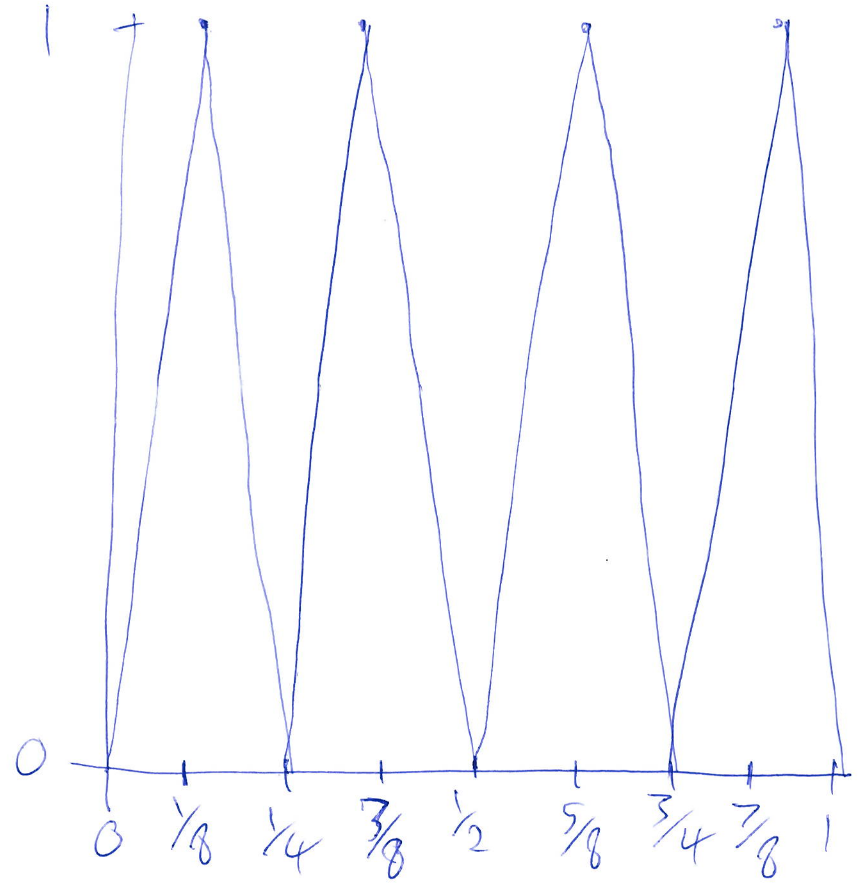
$T^2(x) = 0$ if $x = 0, \frac{1}{2}, 1$

and

$T^2(x) = 1$ if $x = \frac{1}{4}, \frac{3}{4}$

$n=3$

T^3



$T^3(x)=0$ if $x=0, 1/4, 1/2, 3/4, 1$

and

$T^3(x)=1$ if $x=1/8, 3/8, 5/8, 7/8$

In general, $T^n(x)=0$ when

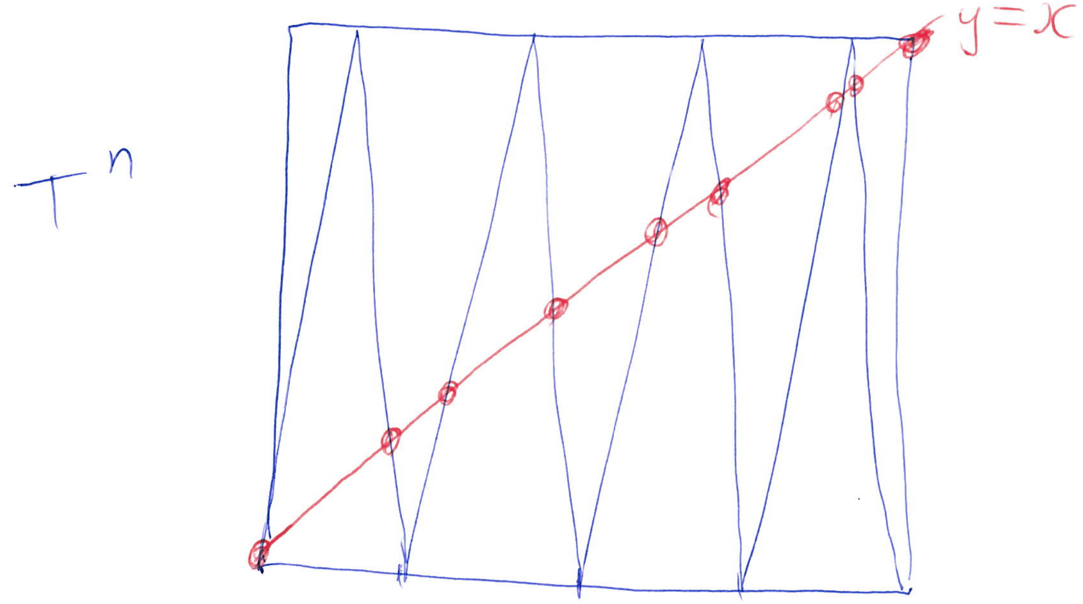
$$x=\frac{2k}{2^n} \text{ for } k=0,1,\dots,2^{n-1}$$

and $T^n(x)=1$ when

$$x=\frac{2k-1}{2^n} \text{ for } k=1,2,\dots,2^{n-1}$$

Note that T^n is linear (and in particular continuous) between successive points in the list

$$\left\{ \frac{j}{2^n} : 0 \leq j \leq 2^n \right\}$$



Therefore T^n has 2^n fixed points.
So T has 2^n points of period n
(note that not all of these will be of <u>least</u> period n).
Precisely one period-n point belongs to each interval of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$

__Examples__ T has e.g. the following periodic points:

Fixed points: 0 and $2/3$

Period-2 points: $\{2/5, 4/5\}$ is a 2-cycle

Period-3 points: $\{2/9, 4/9, 8/9\}$

and $\{2/7, 4/7, 6/7\}$

are 3-cycles.

__Example__ To find a point x of __least__ period 2, suppose $x < 1/2$, then the equation $T^2(x) = x$ must be written as

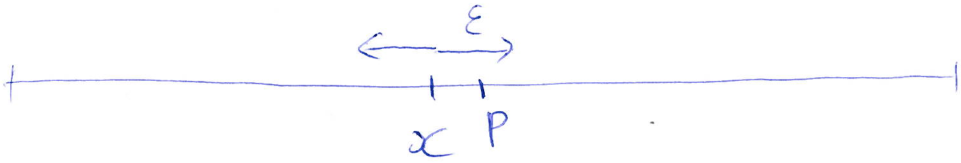
$$T(2x) = x$$

i.e. $2 - 2(2x) = x$

i.e. $2 = 5x$ i.e. $x = 2/5$ is of least period 2

Then $T(2/5) = 4/5$ is also a point of least period 2.

<u>Remark</u> The set of <u>all</u> periodic points (i.e. of all possible periods n) of T is <u>dense</u> in $[0,1]$, i.e. for all $x \in [0,1]$ and for all $\varepsilon > 0$, there exists a periodic point $p \in [0,1]$ such that $|x-p| < \varepsilon$

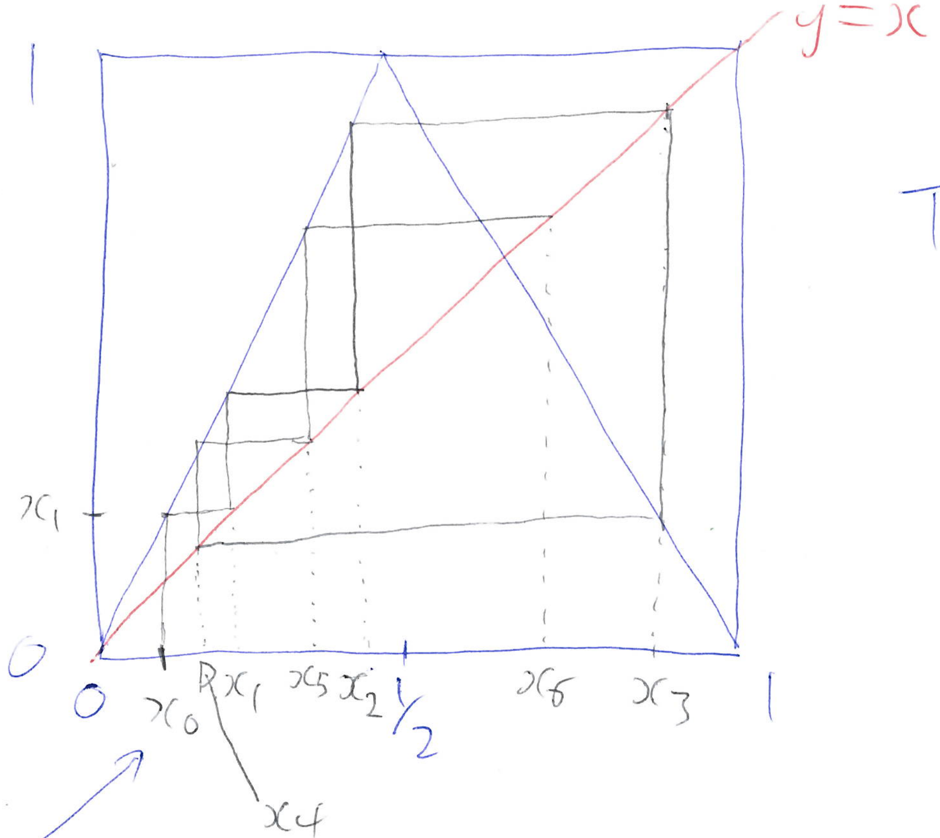


Consequently, the set of <u>all</u> periodic points for the logistic map f_4 ($f_4(x) = 4x(1-x)$) is <u>also</u> dense in $[0,1]$, since there is a topological conjugacy between T and f_4.

Symbolic Coding

We can construct a "left-right itineary" of a point's orbit under T as follows:

For each $x_0 \in [0,1]$, assign a sequence of L's and R's (L = left, R = right), i.e. a sequence where each entry / term in the sequence is either an L or a R, in such a way that the n^{th} term in the sequence is an L if the point $x_n = T^n(x_0)$ is to the left of $1/2$, and is a R if $x_n = T^n(x_0)$ is to the right of $1/2$.



For this choice of x_0 we see that the associated sequence is

$$L\,L\,L\,R\,L\,L\,R\ldots\ldots$$

In this manner we can reformulate questions about the orbits of T (and also the logistic map f_4) into questions about sequences of L's and R's. Our discussion embodies the idea behind:

Symbolic dynamics

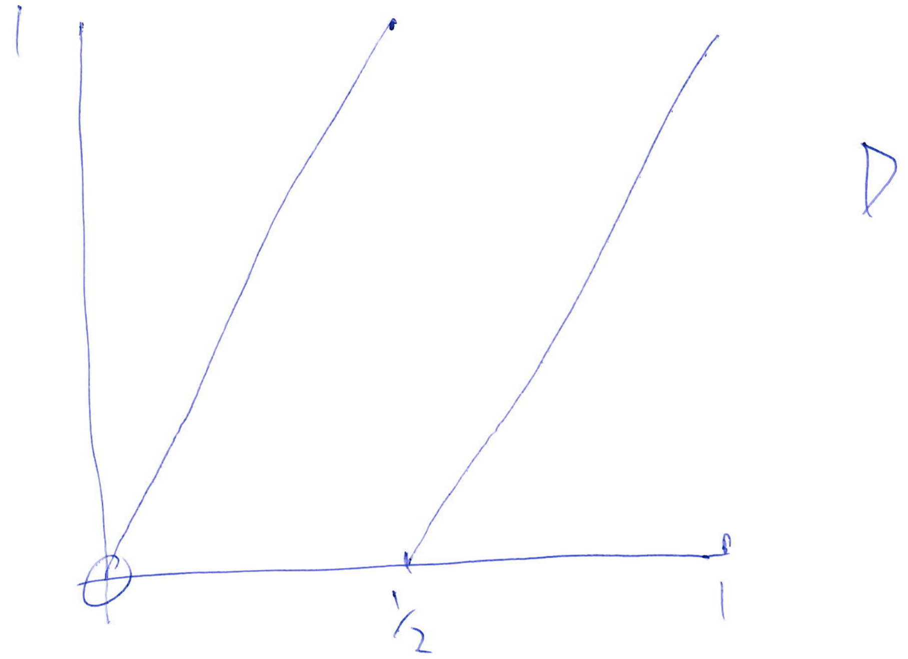
Given a dynamical system, we <u>encode</u> the orbit of a given initial condition x_0 by a sequence of symbols (i.e. an infinite sequence where each term in the sequence is drawn from a given finite set (often the finite set is called an 'alphabet')). This can be used to study the original dynamical system.

Important case : The doubling map

$$D(x) = \begin{cases} 2x & \text{if } x \in [0, \tfrac{1}{2}) \\ 2x - 1 & \text{if } x \in [\tfrac{1}{2}, 1) \end{cases}$$

We think of D as a self-map of $[0,1)$,

i.e. $D : [0,1) \longrightarrow [0,1)$



We can write:

$$D(x) = 2x \pmod 1$$

i.e. $D(x) = \{2x\}$ (where $\{\cdot\}$ means "fractional part")

Remarks. We say that $x = y \pmod 1$ if $x - y$ is an integer

- D has a single fixed point, namely the one at $x = 0$

(note $1 \notin [0,1)$, so 1 is not a fixed point)

- There is a single 2-cycle $\{1/3, 2/3\}$
- There are two 3-cycles, namely $\{1/7, 2/7, 4/7\}$, and $\{3/7, 6/7, 5/7\}$
- In general, a point x has period n if $D^n(x) = x$

i.e. $2^n x = x \pmod 1$

i.e. $2^n x - x = p$ where $p \in \mathbb{Z}$

i.e. $x = \dfrac{p}{2^n - 1}$

Thus the set of period-n points is

$$\mathrm{Per}(n) = \left\{ \frac{p}{2^n - 1} \ : \ p = 0, 1, \ldots, 2^n - 2 \right\}$$

In particular there are precisely $2^n - 1$ points of period n

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If we use symbols 0 and 1 in our symbolic coding (instead of "L" and "R") then for example the point $1/5$, which has least period 4,

$$\frac{1}{5} = \frac{3}{15} = \frac{3}{2^4 - 1}$$

The corresponding 4-cycle is $\{1/5, 2/5, 4/5, 3/5\}$

corresponds to the (periodic) symbol sequence

$$\dot{0}0110011 0011 \ldots$$

Let $S = \{(b_1, b_2, b_3, \ldots) : b_i \in \{0,1\}$ for all $i \in \mathbb{N}\}$.

Define $\sigma : S \longrightarrow S$ by

$$\sigma(b_1 b_2 b_3 \ldots) = b_2 b_3 b_4 \ldots$$

i.e. $\sigma((b_i)_{i=1}^{\infty}) = (b_{i+1})_{i=1}^{\infty}$

This is called the <u>shift map</u>

Define $b : [0,1) \longrightarrow S$

$$b(x) = b_1 b_2 b_3 \ldots$$

where $b_i = \begin{cases} 0 & \text{if } D^{i-1}(x) \in [0, 1/2) \\ 1 & \text{if } D^{i-1}(x) \in [1/2, 1) \end{cases}$

It can be shown that if

$$b(x) = b_1 b_2 b_3 \ldots$$

then

$$x = \sum_{i=1}^{\infty} b_i/2^i = b_1/2 + b_2/4 + b_3/8 + \ldots$$

i.e. the "coding sequence" $b_1 b_2 b_3 \ldots$ for x gives a <u>binary expansion</u> (i.e. <u>base-2 expansion</u>) for x.

<u>Example</u> We have seen that

$$b(1/5) = 0011\,0011\,0011\ldots$$

In this case the binary expansion

$$\sum_{i=1}^{\infty} b_i/2^i \quad \text{gives}$$

$$0/2 + 0/4 + 1/8 + 1/16$$

$$+ \frac{0}{32} + \frac{0}{64} + \frac{1}{128} + \frac{1}{256} + \ldots$$

$$= \left(\frac{1}{8} + \frac{1}{16}\right)\left(1 + \frac{1}{2^4} + \left(\frac{1}{2^4}\right)^2 + \left(\frac{1}{2^4}\right)^3 + \ldots\right)$$

$$= 3/16 \sum_{i=0}^{\infty} \left(1/16\right)^i$$

$$= \frac{3}{16} \frac{1}{1 - \frac{1}{16}}$$

$$= \frac{3}{16} \frac{16}{15}$$

$$= \frac{3}{15}$$

$$= \frac{1}{5}$$

<u>Notation</u> For periodic sequences

$b_1 b_2 \ldots b_n . b_1 b_2 \ldots b_n . b_1 b_2 \ldots b_n \ldots$

We shall often use the notation

$\overline{b_1 b_2 \ldots b_n}$ to denote this sequence.

Note that we can introduce a map $h : S \longrightarrow [0,1)$ defined by

$$h(b_1 b_2 b_3 \ldots) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$$

<u>Claim</u> We then have that

$$D \circ h = h \circ \sigma \qquad (*)$$

In other words,

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "h"'] & S \arrow[d, "h"] \\ {[0,1)} \arrow[r, "D"] & {[0,1)} \end{tikzcd}$$

is commutative.

To check the Claim, note that the RHS of $(*)$ is

$$h \circ \sigma (b_1 b_2 b_3 \ldots)$$

$$= h(b_2 b_3 b_4 \ldots)$$

$$= \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \ldots$$

whereas the LHS of $(*)$ is:

$$D \circ h (b_1 b_2 b_3 \ldots)$$

$$= D\left(\frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \frac{b_4}{16} + \ldots\right)$$

$$= \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \ldots$$

$$= \text{RHS of } (*) \quad \checkmark$$

Note that h is not a bijection, since it is not injective (i.e. it is not one-to-one), as we see with the following example:

$$h(0\bar{1}) = h(0111111\ldots)$$

$$= \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \frac{1}{4} \frac{1}{1 - \frac{1}{2}}$$

$$= \frac{1}{4} 2$$

$$= \frac{1}{2}$$

but also

$$h(1\bar{0}) = h(100000\ldots)$$

$$= \frac{1}{2} + \frac{0}{4} + \frac{0}{8} + \frac{0}{16} + \ldots$$

$$= \frac{1}{2}$$

So $h(0\bar{1}) = h(1\bar{0})$,

So h is <u>not</u> injective

(Compare this to the familiar fact that in decimal notation, the number 1 has the two representations $1 = 1 \cdot 0000\ldots$
$= 0 \cdot 9999\ldots$)

Chaos (What might it mean?)

The doubling map has the property that its periodic points are **dense** in $[0,1)$

i.e. For any $x \in [0,1)$ and any $\varepsilon > 0$ there exists a periodic point y with $|x-y| < \varepsilon$

(Recall that rational numbers of the form $\frac{p}{2^n-1}$ are periodic (of period n) under D)

The doubling map D also has the property that there is a <u>dense orbit</u>, i.e. there is a point $x_0 \in [0,1)$ such that

$$\mathcal{O}(x_0) = \{ D^i(x_0) : i = 0, 1, 2, \ldots \}$$

is dense in $[0,1)$

i.e. For any $x \in [0,1)$, and any $\varepsilon > 0$ there exists $i \geq 0$ such that $| D^i(x_0) - x | < \varepsilon$