Queen Mary University of London

School of Mathematical Sciences
Mile End, London E1 4NS · UK

Examiner: Prof. O. Jenkinson

MTH6107 Chaos & Fractals
MID-TERM TEST
Date: November 2024

Complete the following information:

Name	Model Solutions
Student Number (9 digit code)	

The test has SEVEN questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
Total Marks	

Nothing on this page will be marked!

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Question 1.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 12$.

(a) Find all fixed points of f, and determine whether they are attracting or repelling.

(b) Determine the points of least period 2 for f.

[*25 marks*]

Answer 1.

(a) $x = f(x) = x^2 - 12$

$\Rightarrow 0 = x^2 - x - 12 = (x-4)(x+3)$

$\Rightarrow$ fixed points are 4 and -3

Both are repelling : $|f'(4)| = 8 > 1$, $|f'(-3)| = 6 > 1$

(b) $x = f^2(x) = (x^2 - 12)^2 - 12$

$= x^4 - 24x^2 + 132$

$\Rightarrow 0 = x^4 - 24x^2 - x + 132$

$= (x^2 - x - 12)(x^2 + x - 11)$

$\Rightarrow$ Points of least period 2 are roots of $x^2 + x - 11$, in other words the values $\frac{1}{2}(-1 \pm \sqrt{1 + 44})$

$= \frac{1}{2}(-1 \pm 3\sqrt{5})$.

Answer 1. (*Continue*)

Question 2.

Order the integers from 1 to 20 inclusive using Sharkovskii's order.

[*15 marks*]

Answer 2.

$1 \lhd 2 \lhd 4 \lhd 8 \lhd 16$

$\lhd 20 \lhd 12 \lhd 18 \lhd 14 \lhd 10 \lhd 6$

$\lhd 19 \lhd 17 \lhd 15 \lhd 13 \lhd 11$

$\lhd 9 \lhd 7 \lhd 5 \lhd 3$

Answer 2. (*Continue*)

Question 3.

Suppose the diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + \frac{2}{3}\sin x$. Determine the fixed points of f, and determine whether each fixed point is attracting or repelling.

[*15 marks*]

Answer 3.

$$f(x) = x \iff \tfrac{2}{3}\sin x = 0$$

$$\iff \sin x = 0$$

$$\iff x = n\pi \text{ for } n \in \mathbb{Z}$$

f is C^1, with $f'(x) = 1 + \tfrac{2}{3}\cos x$

So $f'(n\pi) = \begin{cases} 5/3 > 1 & \text{if } n \text{ is even} \\ 1/3 \in (0,1) & \text{if } n \text{ is odd} \end{cases}$

So $n\pi$ is attracting if n is odd, and repelling if n is even

Answer 3. (*Continue*)

Question 4.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1, and that the numbers $1, 2, 3, 4, 5$ form a 5-cycle, with derivatives

$$f'(n) = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Determine, with justification, whether the 5-cycle $\{1, 2, 3, 4, 5\}$ is attracting or repelling.

[*15 marks*]

Answer 4.

The multiplier of this 5-cycle is $\frac{1}{1} \cdot 2 \frac{1}{3} 4 \frac{1}{5} = 8/15 \in (0,1)$,

hence it is attracting

Answer 4. (*Continue*)

Question 5.

Suppose $f : \mathbb{R} \to \mathbb{R}$ has precisely 2 fixed points, and precisely 2 points of least period 2. Can f be a diffeomorphism? Justify your answer.

[10 marks]

Answer 5.

f cannot be a diffeomorphism.

If f was a diffeomorphism then it would either (a) be order preserving or (b) order reversing.

In case (a) it could not have any points of least period 2, by a result we proved, and in case (b) it could not have precisely two fixed points, since an order-reversing diffeomorphism was proved to have precisely one fixed point.

Answer 5. (*Continue*)

Question 6.

Suppose $f : \mathbb{R} \to \mathbb{R}$ has precisely 2 fixed points, precisely 2 points of least period 2, precisely 3 points of least period 3, but no points of least period 4.

Can f be continuous? Justify your answer.

[*10 marks*]

Answer 6.

f cannot be continuous, since by Sharkovskii's Theorem, the presence of an orbit of least period - 3 implies it must have an ~~period~~ orbit of least period 4, but this f does not.

Question 7.

Let $f_4 : [0,1] \to [0,1]$ be the logistic map $f_4(x) = 4x(1-x)$, and let $T : [0,1] \to [0,1]$ be the tent map

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2-2x & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Using the fact that $h \circ T = f_4 \circ h$, where $h : [0,1] \to [0,1]$ is the homeomorphism defined by

$$h(x) = \left(\sin\left(\frac{\pi x}{2}\right)\right)^2,$$

show that $x_0 = (\sin(\pi/9))^2$ is a periodic point for f_4, and determine its least period. *[10 marks]*

Answer 7.

x_0 is periodic, of least period 3, since it is the image under h of the point $2/9$, and it is easily checked that $2/9$ has least period 3 for the tent map (its orbit is $\{2/9, 4/9, 8/9\}$)

Answer 7. (*Continue*)