Queen Mary
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Group Theory

Week 8, Lecture 1, 2 & 3

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Revision

Homomorphism of Groups

$$\varphi : G \longrightarrow H$$

φ is a homomorphism if

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

$$\varphi(1_G) = 1_H$$

$$\varphi(g^{-1}) = (\varphi(g))^{-1}$$

Revision
First Isomorphism of Groups

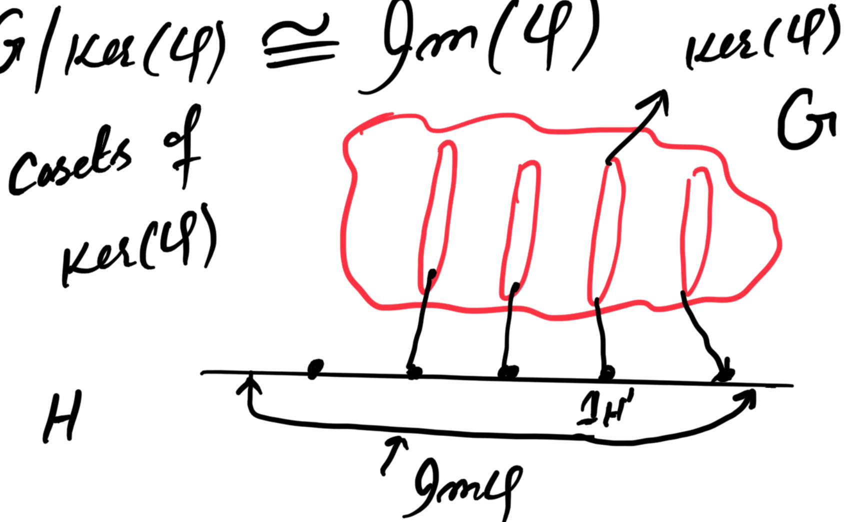
$$\operatorname{Im}\varphi = \{ h \in H : \exists g \in G : \varphi(g) = h \}$$

$$\operatorname{Ker}\varphi = \{ g \in G : \varphi(g) = 1_H \} \trianglelefteq G.$$

Revision

First Isomorphism Theorem:

$$G/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$$



Revision

Second Isomorphism of Groups

Suppose G is a group. $H \trianglelefteq G$, $K \trianglelefteq G$

$K \subseteq H \subseteq G$

i) $K \trianglelefteq H$

$gkg^{-1} \in K \quad \forall g \in G$

$hkh^{-1} \in K$ as $K \trianglelefteq H$

$\forall h \in H$

(ii) $H/K \trianglelefteq G/K$

Revision

Second Isomorphism Theorem:

$\varphi : G/K \longrightarrow G/H = \text{Im}(\varphi)$

$\varphi : Kg \longmapsto Hg$

φ is well-defined

φ is a homomorphism

φ is surjective

$\text{Ker}\,\varphi = H/K$

$$G/K \,/\, H/K \cong G/H$$

$\text{Im}\,\varphi = G/H$

Revision
Third Isomorphism of Groups

G - group, $H \leq G$, $N \trianglelefteq G$

$$HN = \{hn : h \in H, n \in N\} \leq G$$

i) $N \trianglelefteq HN$

ii) $H \cap N \trianglelefteq H$

iii) $H/H \cap N \cong HN/N$

Revision

Example:

$G = \mathcal{U}_{56}$

$H_1 = \{1, 5, 9, 13, 25, 45\}$

$H_2 = \{1, 3, 9, 19, 25, 17\}$

calculate $|H_1 H_2|$

$G = \mathcal{U}_{13},$

$H = \langle 4 \rangle$

$N = \langle 5 \rangle$

Revision

$H \cap N = ?$

$H = \{1, 4, 3, 12, 9, 10\}$

$N = \{1, 5, 12, 8\}$

$H \cap N = \{1, 12\}$ $|H \cap N| = 2 = \mathcal{U}_{13}$

Revision

$$\mathrm{Aut}(G) = \{ \varphi : G \to G,\ \varphi \text{ is a group homomorphism} \}$$

$$= \{ \psi,\ id,\ \xi,\ \psi',\ \ldots \}$$

Automorphism Groups

Definition

An **automorphism** of G is an isomorphism from G to G. The **automorphism group** of G is the set of all automorphisms of G, written as $\mathrm{Aut}(G)$.

Example: $\varphi : G \to G$ $\quad$ $id : x \mapsto x$

φ is a homomorphism

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

$$\varphi(I_G) = I_G \qquad \varphi(g^{-1}) = (\varphi(g))^{-1}$$

$id : g_1 \mapsto g_1$, $\qquad id(g_1) = g_1$

Automorphism Groups

$(\varphi \circ \varphi')(g) = id$ if φ & φ' are inverse of each other.

$\varphi \circ \varphi'(g) = \varphi(\varphi'(g)) = \varphi(\varphi'(g)) = id$

usually we have

$\varphi \circ \varphi^{-1} = id = \varphi^{-1} \circ \varphi$

1) $id : G \to G$, $Aut(G)$ is non-empty

2) $\varphi, \psi \in Aut(G)$, we want to show $(\varphi \circ \psi^{-1}) \in Aut(G)$

$f, g \in G$:

$$(\varphi \circ \psi^{-1})(fg) = \varphi(\psi^{-1}(fg))$$
$$= \varphi(\psi^{-1}(f)\psi^{-1}(g))$$
$$= \varphi(\psi^{-1}(f))\varphi(\psi^{-1}(g))$$

$\therefore \varphi$ is a homomorphism.

$\varphi, \psi \in Aut(G)$

$$(\varphi \circ \psi)(f) = \varphi(\psi(f)) \in Aut(G)$$

Automorphism Groups

Lemma

Suppose $g \in G$, and define $\rho_g : G \to G$ by $\rho_g(h) = ghg^{-1}$. Then ρ_g is an automorphism of G.

Proof:

$\rho_g(h) = ghg^{-1} \in G$, $\quad g, h \in G$, $\rho_g : G \to G$

$= ghg^{-1} = g_2 \in G$

$\Rightarrow \rho_g \in Aut(G)$

Let $h, h_2 \in G$

$\rho_g(hh_2) = ghh_2g^{-1} = ghg^{-1}gh_2g^{-1} = \rho_g(h)\rho_g(h_2)$

Automorphism Groups

Definition

An automorphism of G is called an *inner* automorphism if it has the form ρ_g for some $g \in G$. The **inner automorphism group** of G (written $Inn(G)$) is the set of all inner automorphisms of G.

Definition: Now recall that $Z(G)$ denotes the centre of G, i.e.

$$Z(G) = \{g \in G : gh = hg \text{ for all } h \in \}.$$

<u>Example:</u> $G = C_4 = \{1, g, g^2, g^3\}$

$$\mathrm{Aut}(C_4) = \{ id, \mathrm{Inv} \}$$

$$\begin{aligned} \mathrm{Inv}: z &\mapsto z^3 \\ z^2 &\mapsto z^2 \\ 1 &\mapsto 1 \\ z^3 &\mapsto z \end{aligned}$$

$$\mathrm{Aut}(C_4) \cong C_2$$

$$\mathrm{Aut}(\mathbb{Z})$$

$$\varphi: \mathbb{Z} \to \mathbb{Z} \quad \Big| \; \varphi(a+b) = a+b$$

$$\varphi(a+b) = \varphi(a) + \varphi(b)$$

$$\varphi(a+b) = -(a+b) = -a-b = \varphi(a) + \varphi(b).$$

$$\mathrm{Aut}(\mathbb{Z}) = \{ id, -n \}$$

Automorphism Groups

Proposition

$\mathrm{Aut}(G)$ is a group under composition.

Lemma

Suppose $g \in G$, and define $\rho_g : G \to G$ by $\rho_g(h) = ghg^{-1}$. Then ρ_g is an automorphism of G.

We have seen this above.

Automorphism Groups

Example: Take $G = \mathcal{C}_4 = \{1, z, z^2, z^3\}$. An automorphism ϕ of $\mathcal{C}_4$ is determined by what it does to z (because if $\phi(z) = h$, then $\phi(z^a) = h^a$ for all a). If we take $\phi(z) = 1$ or z^2, then we'll have $\phi(z^2) = 1$; but then ϕ is not injective, because $\phi(1) = 1$ too. So the only possibilities are $\phi(z) = z$ and $\phi(z) = z^3$. And both of these work: the first one is just the identity map, and the second is the automorphism which sends every element to its inverse.

So $\text{Aut}\mathcal{C}_4 = \{id, \phi\}$, where $\phi(z) = z^3$. Observe that $\phi(\phi(z)) = \phi(z^3) = z^9 = z$, so $\phi \circ \phi = id$. So $\text{Aut}\mathcal{C}_4$ has the following Cayley table.

	id	ϕ
id	id	ϕ
ϕ	ϕ	id

Automorphisms Groups

Definition

An automorphism of G is called an **inner automorphism** if it has the form ρ_g for some $g \in G$. The **inner automorphism group** of G (written $\mathrm{Inn}(G)$) is the set of all inner automorphisms of G.

Example:

$$G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$ij = k$

$jk = i$

$ki = j$

$$f_i(1) = i \cdot 1 \cdot i^{-1} = i(1)(-i) = 1$$

$$f_i(-1) = i(-1)i^{-1} = i\cdot(-1)(-i) = -1$$

$$f_i(j) = i \cdot j(-i) = -j$$

$$f_i(-j) = i(-j)(-i) = j$$

$$f_i(k) = i \cdot k(-i) = -k$$

$$f_i(-k) = i(-k)(-i) =$$

$$f_i(j \cdot i) = f_i(j) \cdot f_j(i)$$

Outer Automorphism of Groups

Tutorial:

$$C_G(A) = \{x \in G \mid xa = ax \text{ for all } a \in A\}$$

$$D_{10} = \{1, r, r^2, r^3, r^4, \dots \}$$

Outer Automorphism of Groups

Outer automorphism

The **outer automorphism group** of G is the quotient $\text{Out}G = Aut(G)/Inn(G)$.

$r^2 s = s \cdot r^2$

$r^2(r \cdot s) = (r \cdot s) r^2$

calculate the centre of D_{10}

Normal Subgroups of S_3

Consider S_3, find out the normal subgroup of S_3

$A_3 = \{e, (123), (132)\}$

$S_3 = \{e, (12), (13), (23), (123), (132)\}$

In S_9

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 6 & 1 & 4 & 2 & 7 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 9 \end{pmatrix}$$

i) Disjoint cycles,

ii) orders of permutation

iii) $g = hfh^{-1}$

iv) Conjugate elements

v) $f, h \in A_9$ or not.

$f, g, h \in A_9$

H_1 & H_2

$$H_1 H_2 \neq H_2 H_1$$

D_{10} find the centralizer of D_{10}.

Outer Automorphism of Groups

Example: Let's see an outer automorphism of $\mathcal{D}_8$. Let t denote rotation through 45° clockwise about the centre of the square, and define $\phi : \mathcal{D}_8 \to \mathcal{D}_8$ by $g \mapsto tgt^{-1}$. Then we get

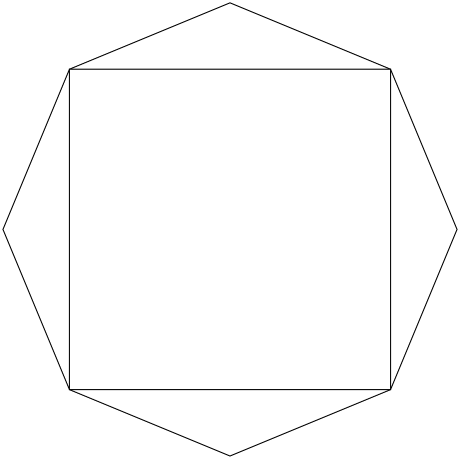
$$\begin{aligned} 1 &\longmapsto 1, & r &\longmapsto r, \\ r^2 &\longmapsto r^2, & r^3 &\longmapsto r^3, \\ s &\longmapsto rs, & rs &\longmapsto r^2 s, \\ r^2 s &\longmapsto r^3 s, & r^3 s &\longmapsto s, \end{aligned}$$

so ϕ is a bijection from $\mathcal{D}_8$ to $\mathcal{D}_8$. The fact that ϕ is a homomorphism is the same as the proof of Lemma 4.7. So $\phi \in Aut(G)$. ϕ cannot be an inner automorphism, because an inner automorphism maps each element to a conjugate element, whereas $\phi(s) = rs \not\sim_{\mathcal{D}_8} s$.

Automorphisms of Groups

In fact $|Aut\mathcal{D}_8| = 8$, and from Theorem 4.8 $|Inn\mathcal{D}_8| = 4$. So $|Out\mathcal{D}_8| = 2$, and hence $Out\mathcal{D}_8 = \{Inn\mathcal{D}_8\, id, Inn\mathcal{D}_8\phi\}$.

To see where ϕ comes from, consider the following picture.



Automorphisms of Groups

We can see that every symmetry of the square gives a symmetry of the octagon, which means that $\mathcal{D}_8 \leq \mathcal{D}_{16}$. In fact, $\mathcal{D}_8 \trianglerighteq \mathcal{D}_{16}$ because $|\mathcal{D}_{16} : \mathcal{D}_8| = 2$. t is an element of $\mathcal{D}_{16}$, so there is an inner automorphism of $\mathcal{D}_{16}$ which maps $g \mapsto tgt^{-1}$ for every g. ϕ is just the restriction of this inner automorphism to $\mathcal{D}_8$.

Exams Style Questions

Question: Let $\phi : G_1 \to G_2$ a homorphism.
(i) If $H_2 \trianglerighteq G_2$, then $\phi^{-1}(H_2) \trianglerighteq G_1$.
(ii) If $H_1 \trianglerighteq G_1$ and ϕ is an epimorphism then $\phi(H_1) \trianglerighteq G_2$.

Proof (i) If $x \in \phi^{-1}(H_2)$ and $a \in G_1$, then $\phi(x) \in H_2$ and so $\phi(axa^{-1}) = \phi(a)\phi(x)\phi(a)^{-1} \in H_2$ since H_2 is normal. We conclude $axa^{-1} \in \phi^{-1}(H_2)$.
(ii) Since H_1 is normal, we have $\phi(a)\phi(H_1)\phi(a)^{-1} \subseteq \phi(H_1)$. Since we assume ϕ is surjective, every $b \in G_2$ can be written as $b = \phi(a)$, $a \in G_1$. Therefore $b\phi(H_1)b^{-1} \in \phi(H_1)$.

Remarks: Note that with the choice $H_2 = \{e\}$ the theorem says that $ker\phi \trianglerighteq G_1$.

QMplus Quiz

Attempt Quiz 8 at QMplus page

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$