WEEK 8 NOTES

1. LAPLACE EQUATIONS ON DISKS AND ANNULI

Recall that the general solution to the Laplace equation in polar coordinates is given by

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty} \Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta). \tag{1.1}$$

We will next apply it to solve boundary value problems

1.1. **Examples of Laplace equations on disks and annuli.**

Consider solutions such that $U(r_*,\theta) = f(\theta)$ and $U(r,\theta)$ well defined at the origin. Observe that the general solution as given by (1.1) are singular at $r = 0$. To avoid this behaviour set $D_0 = 0$ and $D_m = 0$. Hence,

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta), \tag{1.2}$$

where

$$a_0 = C_0 \qquad a_m \equiv A_m C_m \qquad b_m \equiv B_m C_m.$$

The boundary condition at $r = r_*$ gives then

$$U(r,\theta) = f(\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta).$$

This is an example of a Fourier series! The Fourier coefficients can be computed (using the standard method) to be

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \\ b_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta. \end{aligned}$$

Example 1.1. Let

$$\begin{aligned} &\Delta U = 0, \ \text{ in } B_3(0) \\ &U(3,\theta) = 2\cos^2\theta. \end{aligned}$$

Namely $r^* = 3$ and $f(\theta) = 2\cos^2\theta = 1 + \cos 2\theta$.

We have $a_0 = 1$, $b_n = 0$ for all n and $a_n = 0$ except for $n = 2$. Moreover

$$a_2 = \frac{1}{\pi \cdot 3^2}\int_0^{2\pi} \cos^2(2\theta) = \frac{1}{9\pi}\int_0^{2\pi} \frac{\cos 4\theta + 1}{2} = \frac{1}{9}.$$

So $U(r,\theta) = 1 + \frac{1}{9}r^2\cos 2\theta$.

Alternatively, one can compute the coefficients by "observation". Notice that the boundary conditions gives

$$1 + \cos 2\theta = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta)$$

and they have to be equal term by term.

The constant term agrees on both sides gives $a_0 = 1$.

And since the left hand side does not have all the $\cos m\theta$ and $\sin m\theta$ terms except for $\cos 2\theta$ with $m = 2$. So all the a_m and b_m are vanishing except for $m = 2$.

And the coefficients in front of $\cos 2\theta$ agrees, $1 = a_2 \cdot 3^2$. So $a_2 = \frac{1}{9}$.

This gives the same conclusion!

Example 1.2. Let $r_* = 1$ and $f(\theta) = \sin\theta$. One can then evaluate equation (1.2) at $r = 1$to yield

$$U(1, \theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta.$$

As, on the other hand,

$$U(1, \theta) = \sin\theta$$

and the sines and cosines are independent, then by direct inspection one finds that

$$\begin{aligned} &a_0 = 0, \qquad a_m = 0, \\ &b_1 = 1, \qquad b_m = 0, \quad m \neq 1. \end{aligned}$$

So, in this case, the solution takes the simple form

$$U(r, \theta) = r \sin\theta.$$

Example 1.3. Let $r_* = 1$ and $f(\theta) = \cos^2\theta$. Recall the identity

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta.$$

So, in this case we have that

$$\frac{1}{2} + \frac{1}{2}\cos 2\theta = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta,$$

from where direct inspection yields

$$\begin{aligned} &a_0 = \frac{1}{2}, \qquad a_2 = \frac{1}{2}, \qquad a_m = 0, \quad m \neq 0,\, 2, \\ &b_m = 0. \end{aligned}$$

Thus, the solution is given by

$$U(r, \theta) = \frac{1}{2} + \frac{1}{2} r^2 \cos 2\theta.$$

Example 1.4. Now, suppose that the boundary conditions are such that on half of the circle the function takes the constant value U_1 and in the lower part it takes the value U_2. More precisely, one has that

$$f(\theta) = \begin{cases} U_1 & 0 < \theta < \pi \\ U_2 & \pi < \theta < 2\pi. \end{cases}$$

Assume, further for simplicity that $r_* = 1$.

From the theory of Fourier series we have that

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{U_1}{2\pi}\int_0^{\pi} d\theta + \frac{U_2}{2\pi}\int_{\pi}^{2\pi} d\theta = \frac{U_1+U_2}{2}$$
$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi}\cos m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi}\cos m\theta d\theta = 0.$$

However, one also has that

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi}\sin m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi}\sin m\theta d\theta \\ &= -\frac{U_1}{\pi m}\big((-1)^m - 1\big) - \frac{U_2}{\pi m}\big(1-(-1)^m\big) = \frac{(U_1-U_2)\big(1-(-1)^m\big)}{\pi m}. \end{aligned}$$

Hence, the solution to the Laplace equation is given by

$$U(r,\theta) = \frac{U_1+U_2}{2} + \frac{U_1-U_2}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{m}\big(1-(-1)^m\big)\sin m\theta.$$

Observe that the solution only contains terms with m odd. In Exercise 6 of Coursework 7, you will be asked to write the above series in closed form.

Consider now $\Delta U = 0$ in $\Omega = \{r_1 \le r \le r_2\}$ —this type of region is called an annulus (ring) with inner radius r_1 and outer radius r_2. Boundary conditions are then given by

$$U(r_1,\theta) = f(\theta),$$
$$U(r_2,\theta) = g(\theta).$$

In this case one can keep the general solution as the origin is excluded.

Example 1.5. Suppose $\Delta U = 0$ in $\Omega = \{1 \le r \le e\}$ and

$$U(1,\theta) = 4 + (2+e^2)\sin 2\theta$$
$$U(e,\theta) = (2e^2+1)\sin 2\theta.$$

Recall the general solution is of the form

$$U(r,\theta) = \Big(C_0 + D_0\ln r\Big) + \sum_{m=1}^{\infty}\Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m\cos m\theta + B_m\sin m\theta).$$

We must have:

The constant terms satisfies

$$4 = C_0 + D_0\ln 1$$
$$0 = C_0 + D_0\ln e.$$

Thus solving it we get $C_0 = 4$ and $D_0 = -4$.

Next, notice that there are only $\sin$ terms and no $\cos$ terms, we have all $A_m = 0$ for all m.

And since there is only $\sin(2\theta)$ terms, we must have all $B_m = 0$ for $m \neq 2$.

Moreover, we can assume $B_2 = 1$ and solve for C_2, D_2. The coefficients in front of the $\sin 2\theta$ terms gives

$$2 + e^2 = C_2\cdot 1^2 + \frac{D_2}{1^2}$$
$$2e^2 + 1 = C_2\cdot e^2 + \frac{D_2}{e^2}.$$

Solve it we get $C_2 = 2$ and $D_2 = e^2$.

So the solution is

$$U(r,\theta) = 4 - 4\ln r + (2r^2 + \frac{e^2}{r^2})\sin(2\theta).$$

The next examples are the harmonic function in the whole plane $\mathbb{R}^2$ or punctured $\mathbb{R}^2$.

Example 1.6. Find all solutions $U(r,\theta)$ of Laplace equation in the plane

$$\begin{cases} \Delta U = 0, \text{in } \mathbb{R}^2 \\ \lim_{r\to\infty} U(r,\theta) = M_0 \end{cases}.$$

By the boundary condition at infinity, we note that $D_0 = 0$ and $C_m = 0$ for all $m \geq 1$, otherwise the general solutions

$$U(r,\theta) = \left(C_0 + D_0 \ln r\right) + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

will not have a limit as $r \to \infty$.

And by the restriction that U has to be well defined for $r = 0$, we must also have $D_m = 0$ for all $m \geq 1$ from the general solutions.

So the only non-zero term is C_0 and $U(r,\theta) = C_0$. Then the condition that $\lim_{r\to\infty} U(r,\theta) = M_0$ implies $C_0 = M_0$ and so the unique solutions to this problem is

$$U(r,\theta) \equiv M_0.$$

Example 1.7. Find all solutions $U(r,\theta)$ of Laplace equation in the plane with a disk removed

$$\begin{cases} \Delta U = 0, \text{in } \mathbb{R}^2 \setminus \{r < 2\} \\ \lim_{r\to\infty} U(r,\theta) = 0 \\ U(2,\theta) = \cos(4\theta) \end{cases}.$$

By the boundary condition at infinity, we note that $D_0 = 0$ and $C_m = 0$ for all $m \geq 1$, otherwise the general solutions

$$U(r,\theta) = \left(C_0 + D_0 \ln r\right) + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

will not have a limit as $r \to \infty$.

By the condition that $\lim_{r\to\infty} U(r,\theta) = 0$ implies $B_m = 0$ for all m and $A_m = 0$ except for $m = 4$ (as the boundary condition only contains $\cos(4\theta)$ terms). In this situation, we may assume $A_4 = 1$ and the solution becomes

$$U(r,\theta) = \frac{D_4}{r^4}\cos(4\theta).$$

Next, plug in $r = 2$, we get

$$\cos(4\theta) = U(2,\theta) = \frac{D_4}{2^4}\cos(4\theta)$$

and thus $D_4 = 2^4 = 16$. So the solution to this problem is

$$U(r,\theta) = \frac{16}{r^4}\cos(4\theta).$$

1.2. **Poisson's formula.** We have obtained previously the solution to Dirichlet's problem on a disk in the form of the infinite series

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta),$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta, \\ a_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \\ b_n &= \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta. \end{aligned}$$

Remarkably, the previous solution can be written in closed form —i.e. in a way it does not involve an infinite series.

1.2.1. *Some useful facts.* We recall some useful fact that will be used in the following calculation.

Writing trigonometric functions in terms of exponentials. One has the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$. From the above expression it follows that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Geometric series. Recall that for $|x| < 1$ one has that

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

So, in particular

$$x + x^2 + \cdots = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}. \tag{1.3}$$

1.2.2. *Expressing the series solution in closed form.* Substituting the expressions for the Fourier coefficients into the general solutions of Laplace equations on the disk (1.2) one obtains

$$\begin{aligned} U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\left(\int_0^{2\pi} f(\theta')\cos m\theta' d\theta' \cos m\theta + \int_0^{2\pi} f(\theta')\sin m\theta' d\theta' \sin m\theta\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\int_0^{2\pi} f(\theta')\big(\cos m\theta'\cos m\theta + \sin m\theta'\sin m\theta\big)d\theta'. \end{aligned}$$

Recalling that

$$\cos m(\theta - \theta') = \cos m\theta' \cos m\theta + \sin m\theta' \sin m\theta,$$

one gets then

$$\begin{aligned} U(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' + \frac{1}{\pi}\int_0^{2\pi} f(\theta')\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta')d\theta' \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\left(1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta')\right)d\theta'. \end{aligned}$$

Now, rewriting $\cos m(\theta-\theta')$ in terms of integrals one finds that

$$\begin{aligned} 1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta') &= 1+\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\left(e^{im(\theta-\theta')}+e^{-im(\theta-\theta')}\right) \\ &= 1+\sum_{m=1}^{\infty}\left(\frac{re^{i(\theta-\theta')}}{r_*}\right)^m+\sum_{m=1}^{\infty}\left(\frac{re^{-i(\theta-\theta')}}{r_*}\right)^m. \end{aligned}$$

The last two terms are geometric series like in (1.3) series with x given by the expressions in brackets. Accordingly, we can write

$$\begin{aligned} 1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m \cos m(\theta-\theta') &= 1+\frac{(r/r_*)e^{i(\theta-\theta')}}{1-(r/r_*)e^{i(\theta-\theta')}}+\frac{(r/r_*)e^{-i(\theta-\theta')}}{1-(r/r_*)e^{-i(\theta-\theta')}} \\ &= 1+\frac{re^{i(\theta-\theta')}}{r_*-re^{i(\theta-\theta')}}+\frac{re^{-i(\theta-\theta')}}{r_*-re^{-i(\theta-\theta')}} \\ &= \frac{r_*^2-r^2}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}. \end{aligned}$$

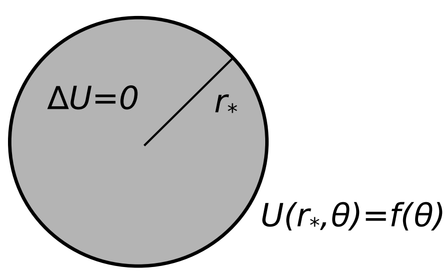
Thus, one has that

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta')(r_*^2-r^2)}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}d\theta',$$

or after some rearrangements

$$U(r,\theta)=\frac{(r_*^2-r^2)}{2\pi}\int_0^{2\pi}\frac{f(\theta')d\theta'}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}. \tag{1.4}$$

The latter is known as *Poisson's formula*. It expresses the solution to the Dirichlet problem on a disk as an integral of the boundary data over the boundary of the disk.



Note. *The term*

$$r_*^2-2rr_*\cos(\theta-\theta')+r^2$$

is, essentially the cosine's law *of trigonometry and gives the distance between a point with polar coordinates* (r, θ) *in the interior of the disk where we want to know the value of* U *and the points* (r_*, θ') *on the boundary of the disk (over which one is integrating).*

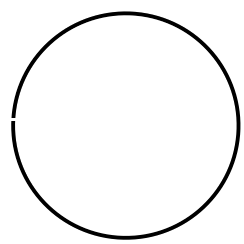
2. THE MEAN VALUE PROPERTY

Several important properties of harmonic functions follow directly from Poisson's formula deduced at the end of last week.

$$U(r,\theta) = \frac{(r_*^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{f(\theta')d\theta'}{r_*^2 - 2rr_* \cos(\theta - \theta') + r^2}. \tag{2.1}$$

In particular, one has the following:

Proposition 2.1 (the first mean value property)**.** *Let* U *be a harmonic function on a disk* Ω*. Then the value of* U *at the centre of the disk is equal to the average of* U *on its circumference.*



Proof. Without loss of generality set the centre of the disk at the origin of the polar coordinates. Then, setting $r = 0$ in Poisson's formula (2.1) one obtains

$$U(0) = \frac{r_*^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')}{r_*^2} d\theta'.$$

The latter can be rewritten as

$$U(0) = \frac{1}{2\pi r_*} \int_0^{2\pi} f(\theta') r_* d\theta'.$$

This is nothing but the average of $f(\theta)$ over the circumference —observe that $2\pi r_\star$ is the value of the circumference while $r_* d\theta'$ is the infinitesimal arc-length. $\square$

Note. *The first mean value property allows one to determine the value of a harmonic function at the centre of the disk without actually having to solve the Laplace equation!*

Example 2.2. For the problem in Example 2.4 last week for a discontinuous temperature on the boundary, a quick calculation gives that

$$\begin{aligned} U(0) &= \frac{1}{2\pi} \int_0^{\pi} U_1 d\theta + \frac{1}{\pi} \int_{\pi}^{2\pi} U_2 d\theta \\ &= \frac{1}{2}(U_1 + U_2). \end{aligned}$$

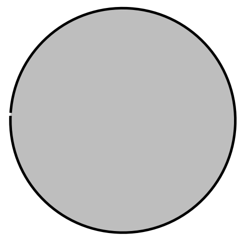
That is the value at the centre is the average of the two different (constant) values at the boundary —this is an intuitive observation.

Notice also that this coincide with the value we get when plugging in $r = 0$ in the solution obtained last week:

$$U(r,\theta) = \frac{U_1 + U_2}{2} + \frac{U_1 - U_2}{\pi} \sum_{m=1}^{\infty} \frac{r^m}{m} \big(1 - (-1)^m\big) \sin m\theta.$$

There is a stronger version of the mean value property:

Proposition 2.3 (the second mean value property)**.** *Let U be a harmonic function on a disk Ω. Then the value of U at the centre of Ω equals the average on the disk.*



Proof. Let $r \le r_*$. The first mean value property then gives that

$$U(0) = \frac{1}{2\pi} \int_0^{2\pi} U(r,\theta) d\theta.$$

Multiplying by $2\pi r$ and integrating from 0 to r_* gives

$$\int_0^{r_*} 2\pi r U(0) dr = \int_0^{r_*} \int_0^{2\pi} U(r,\theta) r d\theta dr$$

However,

$$\int_0^{r_*} 2\pi r U(0) dr = 2\pi U(0) \int_0^{r_*} r dr = \pi r_*^2 U(0).$$

Hence,

$$U(0) = \frac{1}{\pi r_*^2} \int_0^{r_*} \int_0^{2\pi} U(r,\theta) r d\theta dr.$$

The above expression gives the value of u over the disk. In particular, πr_*^2 gives the area of the disk. □

Remark 2.4. One can write the second mean value property in the more concise form

$$U(0) = \frac{1}{\pi r_*^2} \int_{B_{r_*}(0)} U dV,$$

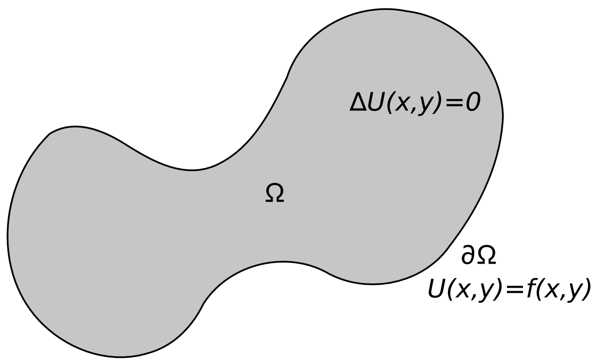
where $\mathcal{B}_{r_*}(0)$ denotes the ball (disk) of radius r_* centred at the origin.

3. The maximum principle

In this section we will discuss the important properties of the maximum and minimum of harmonic functions. These properties have important application —mainly to discuss the uniqueness of solutions to the Laplace and Poisson equation.

We recall some *technical* concepts which will be used in the following discussion:

Open domain. An open domain (i.e. set) is one for which at every point in the set it is possible to have a sufficiently small ball (centred at the point in question) which is contained within the set. In particular, an open domain does not include its boundary.
Connected domain. A connected domain is one which consists only of one piece. More precisely, given two arbitrary points in a connected set, it is always possible to find a curve connecting the two points which is completely contained in the set.



The first result of this section is the following:

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its maximum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x,y)$ is constant for all $(x,y) \in \Omega$.*

Proof. Since $(x_\star, y_\star) \in \Omega$ and Ω is open, we can find $r > 0$ such that $\mathcal{B}_r(x_\star, y_\star) \subset \Omega$. By the mean value property we have that

$$U(x_\star, y_\star) = \frac{1}{\pi r^2} \int_{\mathcal{B}_r(x_\star, y_\star)} U(\underline{x}) d\underline{x}.$$

Since $U(x_\star, y_\star) \geq U(x,y)$ for all $(x,y) \in \Omega$ (it is a maximum!), then the only way to satisfy the mean value property is to have

$$U(x,y) = U(x_\star, y_\star) \qquad \text{for all } (x,y) \in \mathcal{B}_r(x_\star, y_\star).$$

Now, take any point $(x_n, y_n) \in \Omega$. We want to show that $U(x_n, y_n) = U(x_\star, y_\star)$. For this, we connect $(x_\star, y_\star)$ and (x_n, y_n) with a continuous curve that is covered by intersecting balls $\mathcal{B}_{r_0}(x_i, y_i)$, $2r_0 < r$, in such a way that

$$|(x_{i+1}, y_{i+1}) - (x_i, y_i)| < r_0, \qquad \text{for} \quad i = 0, \ldots, n-1.$$

By the first step in the proof one already knows that $U(x_1, y_1) = U(x_\star, y_\star)$. So, repeating the argument we obtain that

$$U(x_\star, y_\star) = U(x_i, y_i) \qquad \text{for} \quad i = 1, \ldots, n.$$

As the domain is connected then any point in it can be joined to $(x_\star, y_\star)$ by means of a curve completely contained in Ω. Thus, The argument used before shows that $U(x,y)$ must be constant throughout Ω. $\square$

Changing U to $-U$ in the previous argument one obtains the following:

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its minimum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x, y)$ is constant for all $(x, y) \in \Omega$.*

Combining the above results one obtains the main result of this section:

Theorem 3.3 (the maximum/minimum principle)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Then U attains its maximum and minimum values on the boundary $\partial\Omega$.*

Note. *In particular, if U is constant on $\partial\Omega$, then it is also constant on Ω.*

Example 3.4. Suppose U is harmonic on the disk of radius 4 with boundary conditions $U(4, \theta) = 4 + 4\cos^2\theta$. Determine the maximum/minimum values of U in the disc of radius r and compute the value of U at the origin.

Notice $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$, we have $U(4, \theta) = 4 + 2\cos 2\theta + 2 = 6 + 2\cos 2\theta$.

By the mean value property

$$U(0) = \int_0^{2\pi} U(4, \theta) d\theta = 6.$$

By the maximum principle and the fact that $-1 \leq \cos\theta \leq 1$. we have

$$\begin{aligned} U_{max} =& 6 + 2 = 8 \\ U_{min} =& 6 - 2 = 4. \end{aligned}$$

Example 3.5. For a harmonic function on the annular region (see Problem set 8 Question 3)

$$\Omega = \{\frac{1}{2} < r < 2\}$$

satisfying the boundary conditions given by

$$\begin{aligned} &U(\frac{1}{2}, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta, \\ &U(2, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta. \end{aligned}$$

Without solving it as in the problem sheets. We can show that $-17 \leq U \leq 51$ on the whole Ω.

Indeed, this follows from the maximum principle and the fact that $-1 \leq \cos 2\theta, \sin 2\theta \leq 1$.

$$\begin{aligned} U \geq& 17 - 17 - 17 = -17 \\ U \leq& 17 + 17 + 17 = 51. \end{aligned}$$