1. The principal part is

$$u_{xx} + 8u_{xy}$$

$a u_{xx} + 2b u_{xy} + c u_{yy} = 0$
with $a=1$, $b=4$, $c=0$

using the new coodirates

$$\begin{cases} x' = x \\ y' = \frac{-4}{1}x + y = -4x + y \end{cases}$$

then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x}$$

$$= \frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y}$$

$$= 0 + \frac{\partial}{\partial y'}$$

$$= \frac{\partial}{\partial y'}$$

So the principal part becomes

$$u_{xx} + 8u_{xy} = \left(\frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}\right)\left(\frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}\right)u + 8\left(\frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}\right)\frac{\partial}{\partial y'}u$$

$$= \left(\frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}\right)(u_{x'} - 4u_{y'}) + 8\left(\frac{\partial}{\partial x'} - 4\frac{\partial}{\partial y'}\right)u_{y'}$$

$$= u_{x'x'} - 4u_{x'y'} - 4u_{x'y'} + 16u_{y'y'} + 8u_{x'y'} - 32u_{y'y'}$$

$$= u_{x'x'} - 16u_{y'y'}$$

and the equation is hyperbolic.

2. Integrate both sides with respect to x,

get $u_y = xy + f(y)$

Integrate again with respect to y.

get $u(x,y) = \frac{xy^2}{2} + F(y) + G(x)$

for any differentiable function F, G.

3. The charateristic curve is

$\frac{dt}{dx} = -2$, $t = -2x + C$, $t + 2x = C$

Along the chrateristic curves, the PDE becomes

$\frac{d}{dx} u(x, t(x)) = e^{-u}$

$e^{u} \cdot u' = 1$

$(e^{u})' = 1$

$e^{u} = x + f(c)$

$u = \ln[x + f(c)]$

The general solution is

$u(x,t) = \ln[x + f(t+2x)]$

when $t=0$, we have $2x=c$, $x=\frac{c}{2}$

$$0 = \ln[x+f(t+2x)] = \ln[\tfrac{c}{2}+f(c)]$$

$$e^0 = \frac{c}{2}+f(c)$$

$$f(c) = e^0 - \frac{c}{2} = 1-\frac{c}{2}$$

plug into the general solution, we get

$$u(x,t) = \ln[x+1-\tfrac{t+2x}{2}]$$

$$= \ln[1-\tfrac{t}{2}]$$

4. The general solution is

$$u(x,t) = F(x+t)+G(x-t)$$

At $x-t=0$, we have $x=t$ and thus

$$4x^2 = F(2x)+G(0)$$

plug in $x=\frac{s}{2}$ we get $F(s) = 4(\frac{s}{2})^2 - G(0) = s^2 - G(0)$

At $x+t=0$, we have $x=-t$ and thus

$$4x = F(0)+G(2x)$$

plug in $x=\frac{s}{2}$ we get $G(s) = 4\cdot\frac{s}{2} - F(0) = 2s - F(0)$

A $(0,0)$ we have $0 = F(0)+G(0)$.

plug into the general solutions we get

$$u(x,t) = (x+t)^2 - G(0) + 2(x-t) - F(0)$$

$$= (x+t)^2 + 2(x-t).$$

5. First consider solutions of the form

$u(x,t) = X(x)\,T(t)$.

The PDE becomes

$$X\ddot{T} - c^2 X'' T = 0$$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda \in \text{constant}$$

The eigenvalue for X becomes

$$① \begin{cases} X'' = -\lambda X \\ X'(0) = 0, \; X(\pi) = 0 \end{cases}$$

claim: $\lambda \geq 0$.

proof of claim: Multiply equation ① by X and integrate, we got

$$\int_0^\pi X(x) X''(x)\,dx = -\lambda \int_0^\pi |X(x)|^2 dx$$

$$X(x)X'(x)\Big|_0^\pi - \int_0^\pi |X'(x)|^2 dx = -\lambda \int_0^\pi |X(x)|^2 dx$$

$$X(\pi)X'(\pi) - X(0)X'(0) - \int_0^\pi |X'(x)|^2 dx = -\lambda \int_0^\pi |X(x)|^2 dx$$

$$-\int_0^\pi |X'(x)|^2 dx = -\lambda \int_0^\pi |X(x)|^2 dx$$

Thus $\lambda \geq 0$. #

so the algebraic equation of ① is

$$x^2 = -\lambda,$$

it has 2 roots $x = i\sqrt{\lambda}, -i\sqrt{\lambda}$

The general solution for X is

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

$X'(0) = 0$ implies

$$0 = -C_1 \sqrt{\lambda} \sin 0 + C_2 \sqrt{\lambda} \cos 0$$

Thus $C_2 = 0$

$X(\pi) = 0$ implies

$$\cos(\sqrt{\lambda}\pi) = 0, \quad \sqrt{\lambda}\pi = \tfrac{1}{2}\pi + n\pi, \; n = 1, 2, \cdots$$

Thus eigen values and eigenfunctions are

$$\lambda_n = \left(\tfrac{1}{2}+n\right)^2, \quad X_n(x) = \cos\left[\left(\tfrac{1}{2}+n\right)x\right]$$

Given λ_n, we solve for T and get

$$T_n(t) = a_n \cos\left[\left(\tfrac{1}{2}+n\right)ct\right] + b_n \sin\left[\left(\tfrac{1}{2}+n\right)ct\right]$$

The general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left[\left(\tfrac{1}{2}+n\right)x\right]\cos\left[\left(\tfrac{1}{2}+n\right)ct\right] + b_n \cos\left[\left(\tfrac{1}{2}+n\right)x\right]\sin\left[\left(\tfrac{1}{2}+n\right)ct\right]$$

Differentiate with respect to t we get

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \left(\tfrac{1}{2}+n\right) \cdot c \cdot \cos\left[\left(\tfrac{1}{2}+n\right)x\right]$$

we have by the initial conditions that

$$0 = \sum_{n=1}^{\infty} a_n \cos\left[\left(\tfrac{1}{2}+n\right)x\right],$$

$$-12c \cdot \cos\left(\tfrac{3}{2}x\right) = \sum_{n=1}^{\infty} b_n \left(\tfrac{1}{2}+n\right) \cdot c \cdot \cos\left[\left(\tfrac{1}{2}+n\right)x\right]$$

We observe that $a_n = 0$ for all n, $b_n = 0$ except for $n = 1$,

and $-12c = b_1 \cdot \left(\tfrac{1}{2}+1\right) \cdot c$, $b_1 = -8$

so the solution is

$$u(x,t) = -8 \cos\left(\tfrac{3}{2}x\right) \sin\left(\tfrac{3}{2}ct\right)$$