Equivalence Relation.

Revision

H a Subgroup of a group G.

Define a relation

$\underbrace{\sim_H \subseteq G \times G}$

$(x, y) \in \sim_H \iff xy^{-1} \in H$.

Equivalence relation:

i) Reflexive: $(x, x) \in \sim_H$

$$e = x \cdot x^{-1} \in H$$

ii) Symmetric

$(x, y) \in \sim_H,$

$(y, x) \in \sim_H.$

iii) Transitive.

$(x, y) \in \sim_H, \quad (y, z) \in \sim_H.$

Question $(x, y) \in M_H$.

M_H defines an equivalence relation on G. $G/M_H = \{ [\ ], [\ ], \dots \}$

D_{10} or D_{12}

look at the subgroups generated by $\langle r^2 \rangle$

$\langle s \rangle$, $\langle r^2 s \rangle$

Write down the coset of $\langle r^2 \rangle$ in D_{10}

$\langle r^2 \rangle = \{ \qquad \}$

Question

$$D_{10} = \{1, r, r^2, r^3, r^4, s, rs, r^2 s, r^3 s, r^4 s\}$$

$H =$

$$\langle r^2 \rangle = \{e, r^2, r^4, r, r^3\}$$

$$= \{e, r, r^2, r^3, r^4\}$$

$$(r^2)^2 = r^4$$

$$(r^2)^3 = r^6 = r$$

$$(r^2)^4 = r^8 = r^3$$

$$(r^2)^5 = e$$

$$Hs = \{s, rs, r^2 s, r^3 s, r^4 s\}$$

$$H = \{e, r^2, r^4, r, r^3\}.$$

$$\langle r^2 s \rangle =$$

Question

Write down the coset.

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R}^*, \ b \in \mathbb{R} \right\}$$

$$\leq SL_2(\mathbb{R},$$

$$A = \begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \frac{1}{a_1} & -b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{a_1} & -b_1 \\ 0 & a_1 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ 0 & 1/a_2 \end{pmatrix}$$

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Question $< SL_2(\mathbb{R})$

$$AB = \begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & \frac{1}{a_2} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 + 0, & a_1 b_2 + \frac{b_1}{a_2} \\ 0 + 0, & 0 + \frac{1}{a_1 a_2} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 & \boxed{a_1 b_2 + \frac{b_1}{a_2}} \\ 0 & \frac{1}{a_1 a_2} \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

Question

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(90^\circ) & \\ & \cos(90^\circ) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} = I$$

$$\{(1, 2\,3)\}$$

$$f \sim_{S_n} g$$

$$\exists\, h,\ h f h^{-1} = g$$

$$S_9$$

Question

$$f=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 6 & 1 & 4 & 2 & 7 \end{pmatrix}$$

$$h=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 9 \end{pmatrix}$$

$$h=(14362)(58)$$

i) Disjoint cycle of f.

$$\boxed{\begin{aligned} f &= (1356)(28)(497) \\ &= \underset{4}{(1356)}\underset{3}{(497)}\underset{2}{(28)} \end{aligned}} \in S_9$$

$$h=\underset{5}{(14362)}\underset{2}{(58)}=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 9 \end{pmatrix}$$

$$g=\overset{4}{(4682)}\overset{2}{(15)}\overset{3}{(397)}$$

$$= h \cdot g \cdot h^{-1}$$

Question

$f \quad g \quad h$

Order: $12, \; 12, \; 10$

$f \cdot g, \; g \cdot f,$

iii) conjugate

f & g are conjugate.

iv) $f, g, h \in A_9$

Remember that $f, g \in A_9$ if we ll be able to write down f and g as even no of Transpositions.

$$f = (1\,3\,5\,6)(2\,8)(4\,9\,7)$$

Question

$$h = (14362)(58)$$

$$g = (4682)(15)(397)$$

We keep the cycle of f and use h to build g

$f \sim_{S_9} g, \Rightarrow \exists h$

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$$hfh^{-1} = g$$

Centraliser Definition

G is a group and $A \subseteq G$

$$C_G(A) = \{ g \in G \mid \forall a \in A : ga = ag \}$$

Question

Consider D_8:

Consider r^3 in D_8

$$D_8 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

$r \cdot r^3 = r^4 = r^3 \cdot r$

r^3 commute with all rotations.

but

$\rightarrow r^3 \cdot s = r^2 \cdot r \cdot s = r^3 \cdot \underline{s \cdot r} \cdot r^{-1}$

$= r^3 \cdot r^{-1} \cdot s \cdot r^{-1}$

$= r^2 \cdot s \cdot \underline{r^{-1}}$

$= r^3 \cdot s \cdot \underline{r^3}$

$= r^3 \cdot \underline{s \cdot r} \cdot r^2$

$= r^3 \cdot \underline{r^{-1} \cdot s} \cdot r^2$

$= r^2 \cdot s \cdot r^2$

$= r^2 \cdot \underline{s \cdot r} \cdot r$

$= r^2 \cdot r^{-1} s \cdot r$

$= r \cdot s \cdot r$

$r^3 = r^{-1}$ as

$r^4 = 1$

$r^3 = r^{-1}$

$$= r r^{-1} s = s$$

Question

Lets Try r^2

$$r^2 \cdot s = r \cdot r \cdot s \cdot r \cdot r^{-1}$$
$$= r^2 \cdot r^{-1} \cdot s \cdot r^{-1}$$
$$= r\, s \cdot r^{-1} = r s r^3$$
$$= r\, s \cdot r \cdot r^2$$
$$= r \cdot r^{-1} s \cdot r^2$$
$$= s \cdot r \cdot r = s r^2$$

$$\boxed{r^2 s = s r^2}$$

$$r^2 \cdot sr = r^2 \cdot r^{-1} s$$
$$= r \cdot s \cdot r \cdot r^{-1}$$
$$= r \cdot r^{-1} s \cdot r^{-1}$$
$$= s \cdot r^{-1}$$
$$= s \cdot r^3$$
$$= sr \cdot r^2$$

$$\boxed{r^2 \cdot sr = sr \cdot r^2}$$