Queen Mary
University of London

JM-TX-IR (Campus-M)

Group Theory

Week 6, Lecture 1, 2 & 3

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Assessment or Mid-Term is in
(11th Nov)
Week 8 - Monday 10:00am
— Friday 5:00pm
(15th Nov)

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No Mid-Term in Week-7

Revision: Commutator subgroups

$[f, g] = fgf^{-1}g^{-1}$ – commutator

* G - abelian $\quad [f, g] = 1$

$G' = \{1\}$

Example: $G = D_8$

$D_8' = \{1, r^2\}$

Revision: Commutator subgroups

Proposition: For any group G, $G' \trianglelefteq G$

observation: G/G' is abelian

G/G' is group.

Homomorphisms

Definition

Suppose G, H are groups.

- A **homomorphism** from G to H is a function $\phi : G \to H$ such that $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in G$.
- An **isomorphism** from G to H is a homomorphism which is also a bijection.
- G, H are **isomorphic** (written $G \cong H$) if there is at least one isomorphism from G to H.

$\varphi(f \circ g) = \varphi f \cdot \varphi g$, for all $f, g \in G$.

$\psi : H \to G \quad \psi(h_1 \cdot h_2) = \psi(h_1) \cdot \psi(h_2)$

$G \cong H$

Homorphisms

Examples: 1) G, H are groups,

$\varphi : g \longmapsto 1$, for all $g \in G$

$\varphi : G \longrightarrow H$

Trivial Homomorphism

$\varphi(g) = 1$

$\varphi(g_1 g_2) = \varphi(g_3) = 1 = 1 \cdot 1 = \varphi(g_1) \cdot \varphi(g_2)$

$GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$

$\det(g_1 g_2) = \det(g_1) \cdot \det(g_2)$

Homorphisms

Examples: 2) $H \leq G$, $i : H \to G$

$i(h) = h$

Inclusion Homomorphism.

3) $N \trianglelefteq G$, $\pi : G \longrightarrow G/N$

$g \longmapsto Ng$

$\pi(g_1 g_2) = N g_1 g_2$

$= (N g_1)(N g_2) = \pi(g_1)\pi(g_2)$

Quotient Homomorphism.

Homomorphisms

$\varphi : G \to H$

Lemma

Suppose G, H are groups and $\phi : G \to H$ is a homomorphism. Then $\phi(1) = 1$, and $\phi(g^{-1}) = (\phi(g))^{-1}$ for every $g \in G$.

Exercise

If $\phi : G \to H$ is an isomorphism then $\phi^{-1} : H \to G$ is an isomorphism too.

proof: $\varphi(1_G) = 1_H$

$\forall g \quad \varphi(g^{-1}) = (\varphi(g))^{-1}$

$(\varphi(g))(\varphi(g^{-1})) = 1_H$

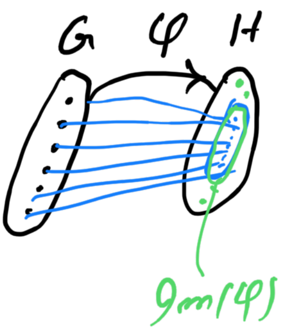
$(\varphi(g^{-1})) \cdot (\varphi(g)) = 1_H$

Homomorphisms

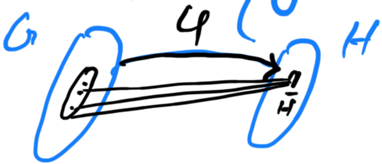
Definition

Suppose $\phi : G \to H$ is a group homomorphism. The **image** of ϕ is the set $\text{im}\phi = \{\phi(g) \mid g \in G\}$. The **kernel** of ϕ is the set $\ker\phi = \{g \in G : \phi(g) = 1\}$.

$$\text{Im}\,\varphi = \{\varphi(g) : g \in G\}$$



$$\text{Ker}\,\varphi = \{g \in G : \varphi(g) = 1\}$$



Homomorphisms

Examples: • The Trivial homomorphism.

$$T : G \to H$$

$$\mathrm{Im}\, T = \{1_H\} \qquad \ker T = G$$

• $H \le G$ $\qquad i : H \to G$

$$i(h) = h$$

$$i : H \to G$$

$$\mathrm{Im}\, i = H$$

$$\ker i = \{1_h\}$$

Homomorphisms

Lemma

Suppose $\phi : G \to H$ is a group homomorphism. Then ϕ is injective if and only if $\ker\phi = \{1\}$.

$\Leftarrow$

$\Rightarrow$

Proof: φ is injective. ✓

If $g \in \mathrm{Ker}(\varphi) \Rightarrow \varphi(g) = 1_H = \varphi(1_G)$

$\Rightarrow \varphi(g) = \varphi(1_G)$

$g = 1_G$

one-one

$\mathrm{Ker}\,\varphi = \{1_G\}$

Homomorphisms

Conversely $\operatorname{Ker}\varphi = \{1\}$

$f, g \in G$, $\varphi(f) = \varphi(g)$

$$\varphi(fg^{-1}) = \varphi(f) \cdot \varphi(g^{-1}) = \varphi(g) \cdot \varphi(g^{-1})$$
$$= 1$$

$fg^{-1} \in \operatorname{Ker}\varphi \Rightarrow fg^{-1} = 1$

$\Rightarrow f = g$

φ is $\longrightarrow$ one-one

Isomorphism Theorem

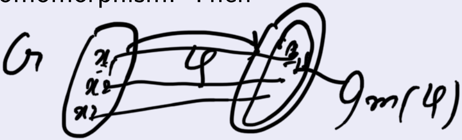
This Theorem is very Important as we use it a lot.

First Isomorphism Theorem

Suppose G, H are groups and $\phi : G \to H$ is a homomorphism. Then

1. $im\phi \leq H$,
2. $ker\phi \trianglerighteq G$, and
3. $G/ker\phi \cong im\phi$.

V. Imp



Proof: 1) $1 \in G$, $\varphi(1) = 1_H \in \operatorname{Im}\varphi$

$\varphi(f), \varphi(g) \in \operatorname{Im}\varphi$

$\varphi(f)\cdot\varphi(g)^{-1} = \varphi(fg^{-1}) = \varphi(f_2) \in \operatorname{Im}\varphi$

$\operatorname{Im}\varphi$ is a Subgroup

Isomorphism Theorem

proof: 2) $\cdot\ 1_G \in Ker\varphi$ $\qquad \varphi(1_G) = 1_H$

$f, g \in Ker\varphi \Rightarrow \varphi(f) = 1_H,\ \varphi(g) = 1_H$

$$\varphi(fg^{-1}) = \varphi(f)\varphi(g^{-1}) = \varphi(f)(\varphi(g))^{-1}$$

$$= 1_H \cdot (1_H)^{-1}$$

$$= 1_H$$

$g \in G,\ n \in Ker(\varphi)$

$$\varphi(gng^{-1}) = \varphi(g)\cdot \underline{\varphi(n)} \cdot \varphi(g^{-1})$$

$$= \varphi(g) \cdot 1_H \cdot \varphi(g)^{-1}$$

$$= 1_H \qquad Ker\varphi \trianglelefteq G$$

First Isomorphism Theorem

Example: Det: $GL_n(F) \to F^\times$ induces an isomorphism $GL_n(F)/SL_n(F) \cong F^\times$

c) Define $\theta : G/\mathrm{Ker}\,\varphi \boxed{\cong} \mathrm{Im}\,\varphi$

$: \mathrm{Ker}\,\varphi \cdot g \longmapsto \varphi(g)$

✓ θ is well-defined.

$f, g \in G$

$\mathrm{Ker}\,\varphi \cdot f = \mathrm{Ker}\,\varphi \cdot g$

by coset lemma

$fg^{-1} \in \mathrm{Ker}\,\varphi$

$\varphi(fg^{-1}) = 1$

✓ θ is a group homomorphism

✓ $\theta((\mathrm{Ker}\,\varphi \cdot f) \cdot (\mathrm{Ker}\,\varphi \cdot g)) = \theta(\mathrm{Ker}\,\varphi \cdot f)\, \theta(\mathrm{Ker}\,\varphi \cdot g)$

$\varphi(f) = \varphi(g)$

θ is Isomorphism.

Second Isomorphism Theorem

$\to \theta$ is one-one & onto

Second Isomorphism Theorem

Suppose G is a group, $H \trianglerighteq G$, $K \trianglerighteq G$ and $K \subseteq H$. Then

1. $K \trianglerighteq H$,
2. $H/K \trianglerighteq G/K$, and
3. $\boxed{(G/K)/(H/K) \cong G/H.}$

i) $K \subseteq H$, $K \leq H$,

$K \trianglelefteq G$

$K \trianglelefteq H$,

$gkg^{-1} \in K$, for all $g \in G$, $k \in K$,

This is true, for all $g \in H$

$K \trianglelefteq H$.

Second Isomorphism Theorem

Proof: 2) $\varphi : G/K \longrightarrow G/H$

$$(G/K)/(H/K) \cong G/H$$ (by first Isomorphism Theorem)

$\varphi : Kg \longmapsto Hg$

* well-define: $Kf = Kg$, by coset Lemma, $fg^{-1} \in K$, $K \subseteq H$, $fg^{-1} \in H$

$Hf = Hg$

* φ is homomorphism:

$$\varphi((Kf)(Kg)) = \varphi(Kfg) = Hfg$$
$$= Hf \cdot Hg$$
$$= \varphi(Kf) \cdot \varphi(Kg)$$

* φ is surjective $Hg \in G/H$ $\varphi(Kg) = Hg$

Third Isomorphism Theorem

For the next isomorphism theorem, recall the product

$$HN = \{hn | h \in H,\ n \in N\}.$$

Third Isomorphism Theorem

Suppose G is a group, $H \leq G$ and $N \trianglerighteq G$. Then

1. $N \trianglerighteq HN$,
2. $H \cap N \trianglerighteq H$, and
3. $H/(H \cap N) \cong (HN)/N$.

Proof:

1) $N \trianglelefteq HN$

$n = 1 \cdot n \in HN,\quad N \subseteq HN$

$\ker(\varphi) = H/K$

$Kg \in G/K$

$Kg \in \ker \varphi$

$\Rightarrow \varphi(Kg) = H \cdot 1$

$= Hg = H \cdot 1$

$\Rightarrow g \in H$

$\ker \varphi = \{Kg \mid g \in H\}$

$= H/K$

Third Isomorphism Theorem

$N \leq HN$

Proof:

$N \trianglelefteq G$, $gng^{-1} \in N \ \forall g \in G$

$HN \subseteq G$.

$gng^{-1} \in N \ \forall g \in HN!$

$N \trianglelefteq HN$

2) $\varphi : H \longrightarrow G/N$

$\varphi(h) = Nh$

$H/H \cap N \cong HN/N$

Third Isomorphism Theorem

φ is a group homomorphism

$$\varphi(hk) = \varphi(h)\varphi(k)$$

$$\varphi(hk) = Nhk = NhNk = \varphi(h)\varphi(k)$$

$\operatorname{Ker}\varphi = H \cap N$ (claim)

$$\operatorname{Ker}\varphi = \{h \in H : \varphi(h) = N \cdot 1\}$$

$$= \{h \in H : N \cdot h = N \cdot 1\}$$

Third Isomorphism Theorem

$$= \{h \in H : h \in N\} = H \cap N$$

$$\Rightarrow$$

$$\operatorname{Im}\varphi = HN/N \text{ (claim)}$$

$$HN/N = \{Nhn : h \in H, n \in N\} = \{Nh : h \in H\}$$

$$Nhn = Nh \text{ by coset Lemma } = \operatorname{Im}\varphi$$

$$hnh^{-1} \in N \text{ (Normal)}$$

Revision: Exams Style Questions

Example: $\varphi: Q_8 \to D_8$ (first Isomorphism)

$\varphi(i) = r^2$

$\varphi(j) = s$

$\varphi(1) = 1$

$\varphi(-1) = r^4 = 1$

$\varphi(-i) = r^2$

$\varphi(j) = s$

$\varphi(-j) = s$

$\varphi(k) = \varphi(i \cdot j) = r^2 s$

$\varphi(-k) = \varphi(-1 \cdot k) = r^2 s$

Check if φ a group homomorphism?

$K = \operatorname{Ker}\varphi = \{1, -1\}$

Revision: Exams Style Questions

$$\operatorname{Im}\varphi = \{1, r^2, s, r^2 s\}$$

$$Q_8/\ker\varphi \cong \operatorname{Im}\varphi$$

$$\{K1, Ki, Kj, Kk\} \qquad \{1, r^2, s, r^2 s\}$$

Define the isomorphism as given above information.

Exams Style Questions

Question: Let $\phi : G_1 \to G_2$ a homorphism.
(i) If $H_2 \trianglerighteq G_2$, then $\phi^{-1}(H_2) \trianglerighteq G_1$.
(ii) If $H_1 \trianglerighteq G_1$ and ϕ is an epimorphism then $\phi(H_1) \trianglerighteq G_2$.

Proof (i) If $x \in \phi^{-1}(H_2)$ and $a \in G_1$, then $\phi(x) \in H_2$ and so $\phi(axa^{-1}) = \phi(a)\phi(x)\phi(a)^{-1} \in H_2$ since H_2 is normal. We conclude $axa^{-1} \in \phi^{-1}(H_2)$.
(ii) Since H_1 is normal, we have $\phi(a)\phi(H_1)\phi(a)^{-1} \subseteq \phi(H_1)$. Since we assume ϕ is surjective, every $b \in G_2$ can be written as $b = \phi(a)$, $a \in G_1$. Therefore $b\phi(H_1)b^{-1} \in \phi(H_1)$.

Remarks: Note that with the choice $H_2 = \{e\}$ the theorem says that $ker\phi \trianglerighteq G_1$.

QMplus Quiz

Attempt Quiz 6 at QMplus page

(Campus-M) - EV-RN-IQ

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \dots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$