Selected solutions to problem set 6

2. The solution is

$$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} 1 \, ds$$

$$+ \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} (-r)\, dr\, ds$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$- \frac{1}{2c}\int_0^t \frac{(x+ct-cs)^2 - (x-ct+cs)^2}{2}\, ds$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$+ \left.\frac{(x+ct-cs)^3}{12c^2}\right|_0^t + \left.\frac{(x-ct+cs)^3}{12c^2}\right|_0^t$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$+ \frac{x^3}{6c^2} + \frac{(x+ct)^3 + (x-ct)^3}{12c^2}$$

3. The solution is

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} 1 \, dr \, ds$$

$$= \frac{1}{2c}\int_0^t \left[(x+ct-cs) - (x-ct+cs)\right] ds$$

$$= \frac{1}{2c}\int_0^t (2ct - 2cs)\, ds$$

$$= \frac{1}{2c}\cdot \left[2cts - cs^2\right]\Big|_0^t$$

$$= \frac{1}{2c}\left[2ct^2 - ct^2\right]$$

$$= \frac{t^2}{2}$$

(3) 5. The general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + b_n \sin(nx)\sin(nct)$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \sin(nx)\,dx$$

$$= \begin{cases} 0 & , n \neq 1 \\ \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx & , n=1 \end{cases}$$

$$= \begin{cases} 0 & , n \neq 1 \\ 1 & , n=1 \end{cases}$$

So the soltn is

$$u(x,t) = \sin x \cos(ct)$$

6. (1) For $x > 0$,

$F(x) = f(x)$

and becase $-x < 0$,

$F(-x) = f(-(-x)) = f(x) = F(x)$

So F is even.

G is even by the same reason.

The derivative of an even function is always odd.

(2) The solution is

$V(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)]$

$+ \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$

(3) $V_x(x,t) = \frac{1}{2}[F'(x+ct) + F'(x-ct)]$

$+ \frac{1}{2c}G(x+ct) - \frac{1}{2c}G(x-ct)$

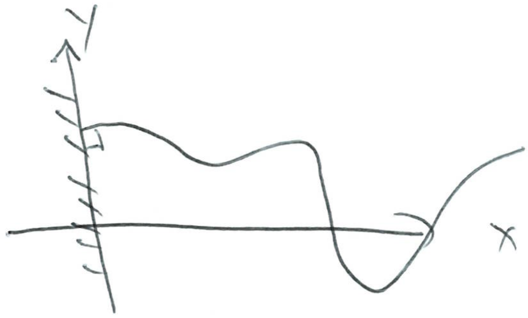
(4) $V_x(0,t) = \frac{1}{2}[F'(ct) + F'(-ct)]$

$+ \frac{1}{2c}[G(ct) - G(-ct)]$

$= 0$

becase F' is odd and G is even.

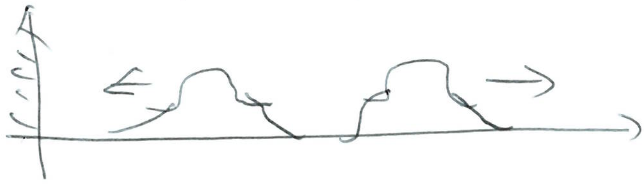
(5)



The string is allowed to move up and down, but always orthognoal to the y axis at the left end.

For an intial bump 

It first separate into 2 bumps



~~A~~ ~~At~~ After some time the left bump hit the "wall" (y-axis) and is reflected. then both bumps are propagating to right

④ 8. • The Energy is

$$E[u](t) = \int_{-\infty}^{\infty} \left[\frac{1}{2} u_t^2 + \frac{1}{2} a^2 u_x^2\right] dx$$

Multiply both sides of equation by u_t, we get

$$u_t \cdot u_{tt} - a^2 u_t u_{xx} - c u_t^2 = 0$$

integrate get

$$\int_{-\infty}^{\infty} u_t \cdot u_{tt} dx - \int_{-\infty}^{\infty} a^2 u_t u_{xx} dx - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

$$\int_{-\infty}^{\infty} \frac{u_t^2}{2} \cdot dx$$

$$\int_{-\infty}^{\infty} \ldots - \int_{-\infty}^{\infty} a^2 u_t u_{xx} - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

By integration by part., we get

$$\int_{-\infty}^{\infty} \frac{(u_t^2)_t}{2} dx - a^2 u_t \cdot u_x \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} a^2 u_{xt} u_x dx - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

thus

(5) $$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{(u_t)^2}{2}dx - 0 + \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{a^2(u_x)^2}{2}dx - c\int_{-\infty}^{\infty}u_t^2 = 0.$$

Namely

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(\frac{u_t^2}{2}+\frac{a^2u_x^2}{2}\right)dx = c\int_{-\infty}^{\infty}u_t^2 \leq 0$$

So the energy is non-increasing because $c<0$.

- Suppose u_1, u_2 are two solutions to $\begin{cases} u_{tt} - a^2u_{xx} - cu_t = F(x), & c<0 \\ u(x,0) = f(x), \ u_t(x,0) = g(x) \end{cases}$

then $u = u_1 - u_2$ is a solution to

$$\begin{cases} u_{tt} - a^2u_{xx} - cu_t = 0, & c<0 \\ u(x,0) = 0, \ u_t(x,0) = 0 \end{cases}$$

(b) We see ~~that~~ from the first part ~~of~~ the question that

$$\frac{d}{dt} E[u](t) = 0,$$

But $E[u](0) = \int 0^2 + 0^2 \, dx = 0$

So $E[u](t) \equiv 0$ for all t.

Namely $u_t \equiv 0$, $u_x \equiv 0$

and so $u \equiv 0$.

we must have $u_1 = u_2$

and the solution to

$$\begin{cases} u_{tt} - a^2 u_{xx} - c u_t = 4(x), & c < 0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \end{cases}$$

is unique.

PS6 Q1:

using the formula (3.6) in Week 6 notes for inhomogeneous problem with

$\psi(x) = \sin x$
$f(x) = 0$ and $c = 1$
$g(x) = \sin x$

we get

$$u(x,t) = \frac{0+0}{2} + \frac{1}{2}\int_{x-t}^{x+t} \sin s\, ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \sin r\, dr\, ds$$

$$= \frac{\cos(x-ct) - \cos(x+ct)}{2} + \frac{1}{2}\int_0^t [\cos(x-t+s) - \cos(x+t-s)]\, ds$$

$$= \frac{\cos(x-ct) - \cos(x+ct)}{2} + \frac{\sin x - \sin(x-t)}{2}$$

$$- \frac{1}{2} \cdot [-\sin x + \sin(x+t)]$$

$$= \frac{\cos(x-ct) - \cos(x+ct)}{2} + \sin x - \frac{\sin(x-ct) + \sin(x+ct)}{2}$$

PS 6 Q4:

For this Question, we can directly apply the formula in section 3 of Week 5 notes

with $L = \pi$

$$f = b\sin x + 2b\sin(2x)$$

$$g = \sin x$$

thus $a_n = \frac{2}{\pi}\int_0^L [b\sin x + 2b\sin(2x)]\sin(nx)dx$

$$b_n = \frac{2}{n\pi c}\int_0^L \sin x \sin(nx)\, dx$$

Using the orthogonality of $\sin nx$ and $\sin mx$

we see all $a_n = 0$ except for $n = 1, 2$,

and $a_1 = b$, $a_2 = 2b$

all $b_n = 0$ except for $n = 1$

and $b_1 = \frac{1}{c}$

so

$$u(x,t) = b\cdot \sin x\cos(ct) + 2b\sin(2x)\cos(2ct) + \frac{1}{c}\sin x\sin(ct)$$

PS 6 Q 8:

- we differentiate $E[u](t)$ with respect to t and get

$$\frac{d}{dt} E[u](t)$$

$$= \frac{d}{dt} \frac{1}{2}\int_{-\infty}^{\infty} \left[u_t^2 + \partial^2 u_x^2\right] dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{d}{dt}[u_t^2 + \partial^2 u_x^2] dx$$ ✓ chain rule

$$= \frac{1}{2}\int_{-\infty}^{\infty} [2 \cdot u_t \cdot u_{tt} + 2\partial^2 \cdot u_x \cdot u_{xt}] dx$$

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx + \partial^2 \int_{-\infty}^{\infty} u_x \cdot u_{xt} dx$$ ✓ integration by parts

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx + \partial^2 u_x \cdot u_t \Big|_{x=-\infty}^{\infty} - \partial^2 \int_{-\infty}^{\infty} u_{xx} \cdot u_t dx$$

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx - \int u_t \cdot \partial^2 u_{xx} dx$$ ✓ using compact support.

$$= \int_{-\infty}^{\infty} u_t [u_{tt} - \partial^2 u_{xx}] dx$$

$$= \int_{-\infty}^{\infty} u_t \cdot c u_t$$ ✓ using the PDE $u_{tt} - \partial^2 u_{xx} = c \cdot u_t$

$$= c \cdot \int_{-\infty}^{\infty} u_t^2$$

≤ 0 (because $c < 0$ and $\int_{-\infty}^{\infty} u_t^2 \geq 0$)

so the energy is non increasing.

- For the second part of the question, suppose u_1 and u_2 are 2 solutions to

$$\begin{cases} u_{tt} - \gamma^2 u_{xx} - c u_t = F(x), \quad c < 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases} \quad (**)$$

then $v = u_1 - u_2$ is a solution to

$$\begin{cases} v_{tt} - \gamma^2 v_{xx} - c v_t = 0 \\ v(x,0) = 0, \quad v_t(x,0) = 0 \end{cases}$$

Notice the initial condition gives

$$E[v](0) = \frac{1}{2} \int 0 + \gamma^2 \cdot 0 = 0$$

and the first part shows us $E[v](t)$ is non-increasing.

But because $E[v](t) \geq 0$ for all t,

so we must have $E[v](t) \equiv 0$

Namely $v_x \equiv 0$ and $v_t \equiv 0$.

$$v(x,t) = \int_0^t v_t(x,s)\,ds + v(x,0)$$

$$= 0 + 0 = 0$$

So $v \equiv 0$ and thus $u_1 \equiv u_2$

and the solution to $(**)$ is unique.

Solution to <u>PS 6 Q 7</u>:

Show that the energy of the solution $u(x,t)$ to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \geq 0, t \geq 0 \\ u(0,t) = 0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \end{cases} \quad (\text{¥})$$

for the wave equation on <u>half line</u> is <u>conserved</u>.

Assuming f, g are compact support, use the conservation of energy to show uniqueness of solutions.

The Energy $E[u](t) = \int_0^\infty \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right) dx$

has time derivative

$$\frac{d}{dt} E[u](t) = \int_0^\infty \left(\frac{1}{2} \cdot 2u_t u_{tt} + \frac{1}{2} c^2 \cdot 2 u_x u_{xt}\right) dx$$

$$= \int_0^\infty u_t \cdot u_{tt} \, dx + c^2 u_x \cdot u_t \Big|_0^\infty - \int_0^\infty c^2 u_{xx} \cdot u_t \, dx$$

(Integration by parts)

The boundary condition gives $u_t(0,t) = \frac{d}{dt} u(0,t) = 0$

The condition of compact support gives $u_x \cdot u_t$ is zero at ∞.

so $$\frac{d}{dt} E[u](t) = \int_0^\infty u_t \cdot (u_{tt} - c^2 u_{xx})\, dx$$

$$= \int_0^\infty u_t \cdot 0 \, dx$$

by the equation

$$= 0$$

so Energy is conserved.

Next, if u_1, and u_2 are 2 solutions to (**)

then $w = u_1 - u_2$ is a solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , x \geq 0, t \geq 0. \\ u(0,t) = 0 \\ u(x,0) = 0, \ u_t(x,0) = 0 \end{cases}$$

Noticing $E[w](0) = 0$, the preservation of energy implies $E[w](t) \equiv 0$ for all t.

so $w_t \equiv 0$, and thus

$$w(x,t) = \int_0^t w_t(x,t) + w(x,0) = 0 + 0 \equiv 0$$

Thus $u_1 - u_2 \equiv 0$ and $u_1 \equiv u_2$.

The solution is unique.