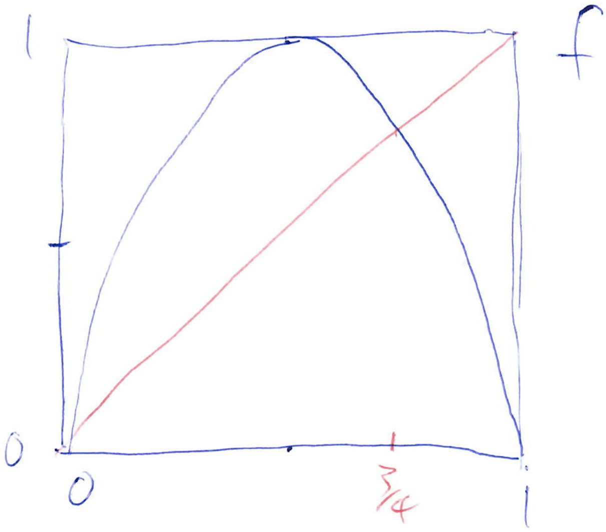
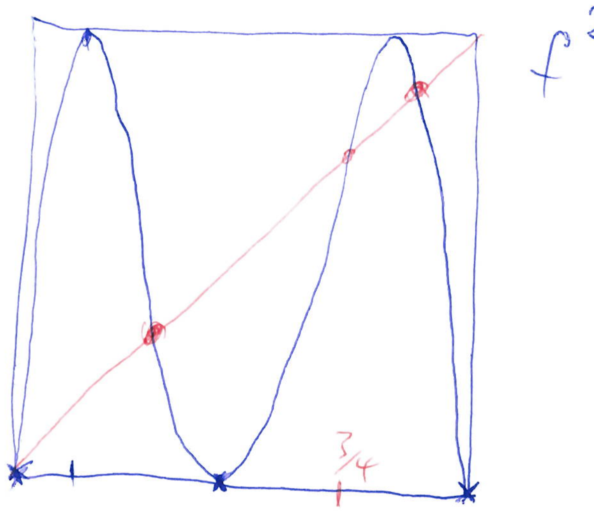
Period-3 points : $f^3(x) = x$

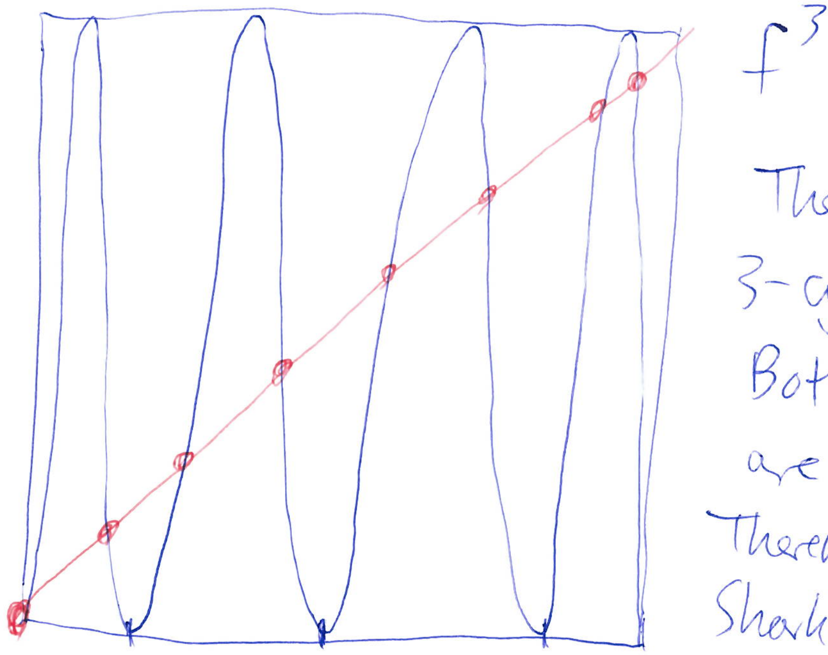
$f^3(x) - x$ } degree-8 polynomial

$\quad\|$

$x(4x-3)(\text{cubic factor})(\text{cubic factor})$







There are two 3-cycles for f. Both 3-cycles are repelling. Therefore, by Sharkovskii's Theorem, there are n-cycles for all $n \in \mathbb{N}$.

Topological conjugacy

Defn If X and Y are intervals in $\mathbb{R}$, we ~~[illegible]~~ say $h : X \to Y$ is a homeomorphism if h is a bijection (i.e. invertible) and both h and h^{-1} are continuous.

Defn Let X and Y be intervals in $\mathbb{R}$. Assume $f : X \to X$ and $g : Y \to Y$. We say that $h : X \to Y$ is a topological conjugacy from f to g if

(i) h is a homeomorphism

(ii) $h \circ f = g \circ h$

Note regarding terminology:

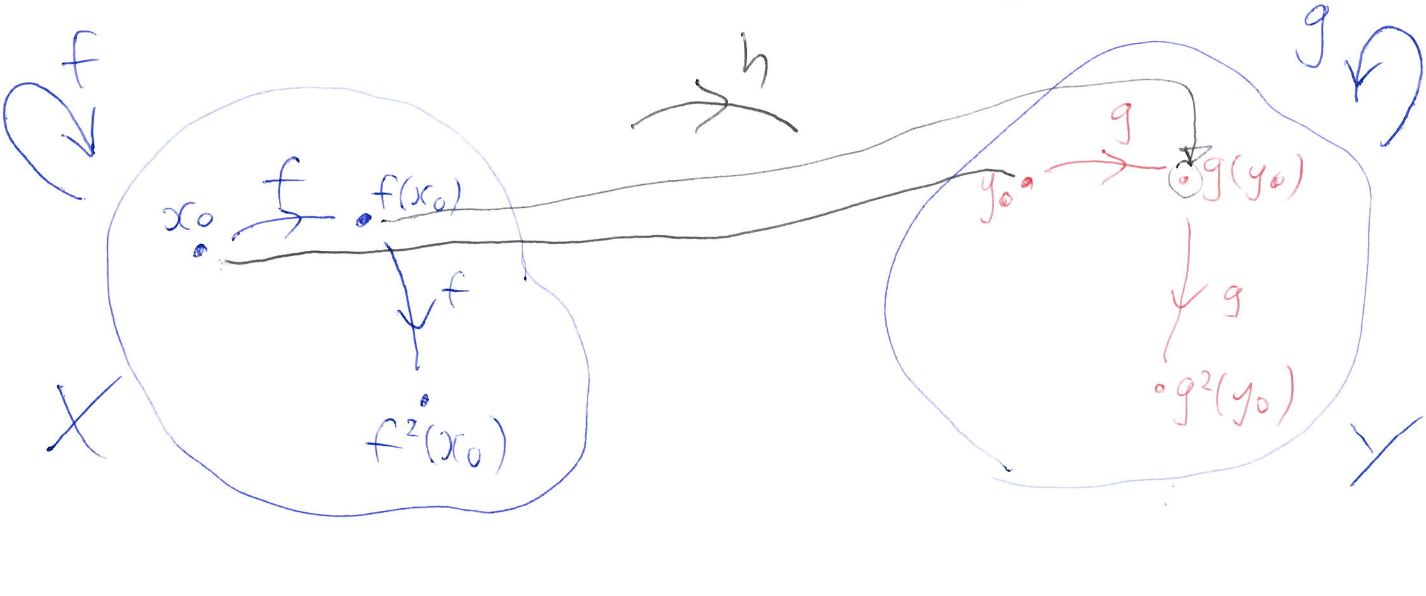
- We also refer to h as a topological conjugacy between f and g

  or between g and f

- We say that f and g are topologically conjugate if an h as above exists

- We also say that f is topologically conjugate to g

Remarks (a) Conjugacy can be viewed as a change of coordinates / variables



So, if we want to study the orbit of x_0 under f in X, we can use h to equivalently study the orbit of $y_0 = h(x_0)$ under g in Y.

(b) The conjugacy equation $\underbrace{h \circ f = g \circ h}$ can be written as

i.e. This means $h(f(x)) = g(h(x))$ for all $x \in X$

$$f = h^{-1} \circ g \circ h$$

or $g = h \circ f \circ h^{-1}$

or $f \circ h^{-1} = h^{-1} \circ g$

[i.e. $h^{-1} \circ g = f \circ h^{-1}$]

(c) Another way of thinking of the conjugacy equation (popular with algebraists) is that it is saying that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

<u>Example</u> We can find a topological conjugacy from the logistic map

$$f_\mu(x) = \mu x(1-x) \qquad (\mu > 0)$$

to $g_c(x) = x^2 + c$, for some suitable value of c (depending on μ).

To see this, we shall view f_μ and g_c as maps $\mathbb{R} \to \mathbb{R}$ (i.e. $X = Y = \mathbb{R}$)

Let's guess that the conjugacy h is of the form $h(x) = \alpha x + \beta$ $(\alpha \neq 0)$ (i.e. assume h is linear / affine)

Such an h is clearly a homeomorphism.

We want to find α, β such that

$$h(f_\mu(x)) = g_c(h(x)) \quad \forall x \in \mathbb{R}$$

i.e. $\alpha f_\mu(x) + \beta = g_c(\alpha x + \beta)$

$$\alpha \mu x(1-x) + \beta = (\alpha x + \beta)^2 + c$$

$$\alpha \mu x - \alpha \mu x^2 + \beta = \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + c$$

Comparing coefficients gives:

x^2 : $\quad -\alpha\mu = \alpha^2 \quad$ i.e. $-\mu = \alpha$

x : $\quad \alpha\mu = 2\alpha\beta \quad$ i.e. $\mu = 2\beta$

i.e. $\beta = \mu/2$

constant : $\quad \beta = \beta^2 + c \quad$ i.e. $c = \beta - \beta^2$

i.e. $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$

So, for $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$ then f_μ and g_c are indeed topologically conjugate, where the topological conjugacy h is given by $h(x) = -\mu x + \mu/2$

e.g. So for example if $\mu = 4$,
i.e. we are considering $f_4(x) = 4x(1-x)$
then this is topologically conjugate to

$$g_{-2}(x) = x^2 - 2$$

because $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$

$$= \frac{4}{2} - \left(\frac{4}{2}\right)^2 = 2 - 4 = -2$$

[Recall that earlier we looked at
$f(x) = 4x(1-x)$, and a few
weeks ago we studied $x \mapsto x^2 - 2$]

<u>Lemma</u> Topological conjugacy is an
equivalence relation.

<u>Proof</u> Exercise. $\square$

Lemma If $h : X \to Y$ is a topological conjugacy between $f : X \to X$ and $g : Y \to Y$ then h is also a topological conjugacy between f^n and g^n, for all $n \in \mathbb{N}$.

Proof We can write the conjugacy equation as $f = h^{-1} \circ g \circ h$.

Then $f^n = f \circ f \circ \ldots \circ f$

$$= (h^{-1} \circ g \circ h) \circ (h^{-1} \circ g \circ h) \circ \ldots \circ (h^{-1} \circ g \circ h)$$

these "cancel" since $h \circ h^{-1} = id =$ identity map

$$= h^{-1} \circ g \circ id \circ g \circ id \circ \ldots \circ id \circ g \circ h$$

$$= h^{-1} \circ g \circ g \circ \ldots \circ g \circ h$$

$$= h^{-1} \circ g^n \circ h$$

and this is a conjugacy equation, so f^n and g^n are topologically conjugate. $\square$

Proposition Topological conjugacy preserves orbits, periodic points, and (least) periods of orbits.

Proof Examine $\mathcal{O}_f(x_0) = \{f^n(x_0) : n \geq 0\}$ (the orbit of x_0 under f).

Then
$$h(\mathcal{O}_f(x_0)) = \{h(f^n(x_0)) : n \geq 0\}$$
$$= \{g^n(h(x_0)) : n \geq 0\} \quad \text{by previous Lemma}$$
$$= \mathcal{O}_g(h(x_0))$$
(the orbit of $y_0 = h(x_0)$ under g)

So we have seen that the image under h of an orbit is itself an orbit.

In particular, a periodic orbit for f is mapped by h to a periodic orbit for g.

Exercise: Check that the (least) period of the orbit is preserved by h. $\square$

Important: The preceding Proposition gives a way of showing that 2 maps f and g are not topologically conjugate.

Corollary: If for some $n \in \mathbb{N}$, the map f has a point of period n, but g does not have a point of period n, then f and g are not topologically conjugate.

Example If f has a fixed point but g does not, then f and g are not topologically conjugate.

Corollary If two maps f, g are topologically conjugate, then for every $n \in \mathbb{N}$, the number n-cycles for f is equal to the number of n-cycles for g.

<u>Example</u> For $\mu = 1+\sqrt{8} \simeq 3.83\ldots$, the logistic map $f_\mu(x) = \mu x(1-x)$ is topologically conjugate $g_c(x) = x^2 + c$ where

$$c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$$

$$= \tfrac{1}{2}(1+\sqrt{8}) - \tfrac{1}{4}(1+\sqrt{8})^2$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{8} - \tfrac{1}{4}(1+2\sqrt{8}+8)$$

$$= -\tfrac{7}{4}$$

It is convenient to study $g_c(x) = x^2 + c$ with $c = -\tfrac{7}{4}$, as it can be shown that a <u>period-3 orbit</u> emerges at this value of c.

The equation $g_c^3(x) = x$ becomes

$$((x^2+c)^2+c)^2+c = x \quad (*)$$

but fixed points of g_c satisfy this equation, in other words

$$g_c(x) - x = x^2 - x + c$$

is a factor of $g_c^3(x) - x$

Factorising $g_c^3(x)-x$ we get that $(*)$ becomes:

$$(x^2-x+c)(x^6+x^5+(3c+1)x^4+\dots) = 0 \quad (**)$$

For $\underline{\text{general}}$ c we we expect complex (not real) roots, however when $c = -7/4$ we find that the LHS of $(**)$ is:

$$(x^2-x+c)\left(x^3 - \frac{x^2}{2} - \frac{9x}{4} - \frac{1}{8}\right)^2$$

The roots of this cubic are real and give a period-3 orbit

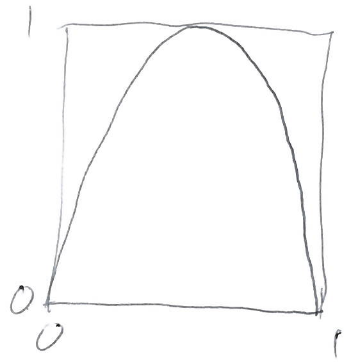
Symbolic Dynamics

Motivating discussion:

We can further analyse the specific logistic map $f_4(x) = 4x(1-x)$

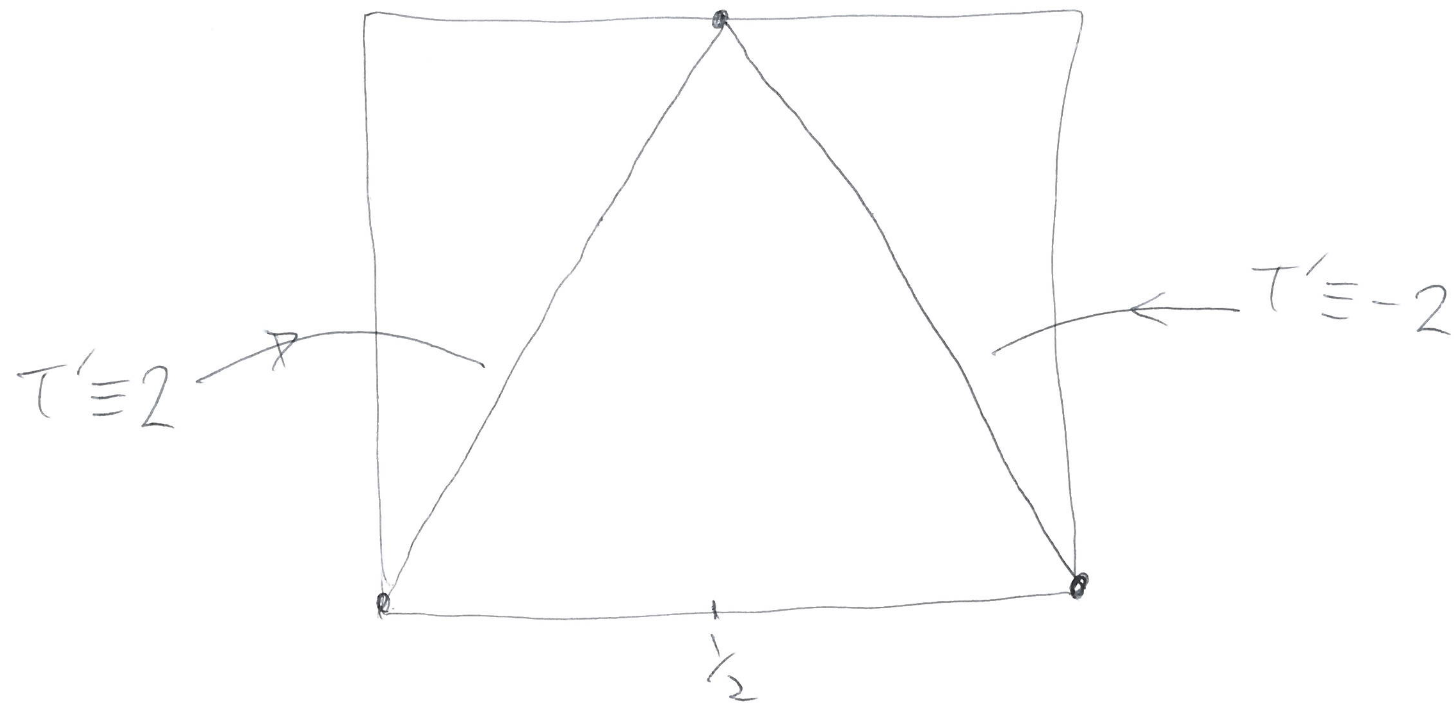
$$f_4 : [0,1] \to [0,1] \quad (\text{i.e. } \mu = 4)$$

using topological conjugacy.



<u>Defn</u> The <u>tent map</u> T is given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x < \tfrac{1}{2} \\ 2-2x & \text{if } \tfrac{1}{2} \le x \le 1 \end{cases}$$

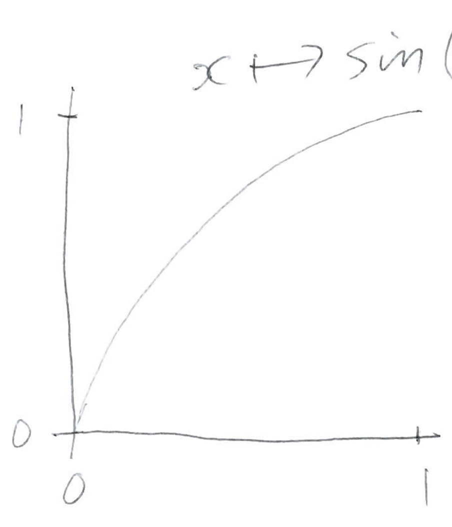


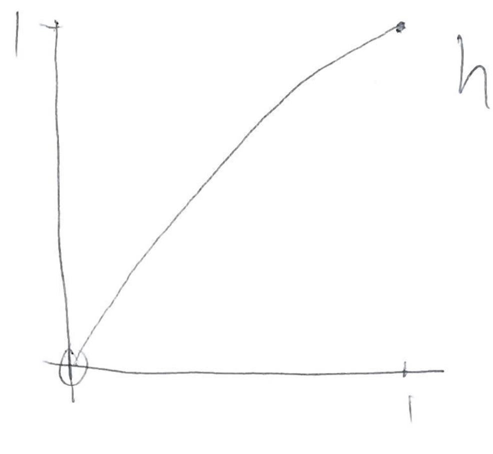
<u>Claim</u> The tent map $T : [0,1] \to [0,1]$ and the logistic map $f_4 : [0,1] \to [0,1]$ are topologically conjugate, with conjugacy map $h : [0,1] \to [0,1]$ defined by

$$h(x) = \left(\sin\left(\frac{\pi x}{2}\right)\right)^2$$

$$\left[= \sin^2\left(\frac{\pi x}{2}\right) \right]$$

<u>Proof of Claim</u> To see this, first note that $h : [0,1] \to [0,1]$ is a homeomorphism. Note that $h(0) = 0$ and $h(1) = 1$, so h is surjective (by the Intermediate Value Theorem) and $h'(x) = 2\sin\left(\frac{\pi x}{2}\right) \cdot \cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2} > 0$ for $x \in (0,1)$, so h is injective. So h is bijective, and h, h^{-1} are continuous, so h is a homeomorphism.





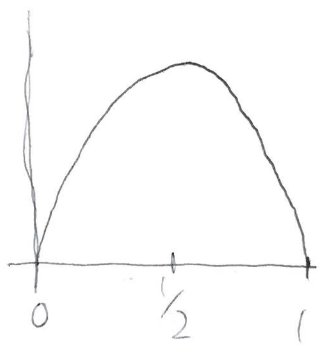
Now we need to check the conjugacy equation $h \circ T = f_4 \circ h$ holds.

Now
$$h \circ T(x) = \begin{cases} \left(\sin\left(\frac{\pi}{2} \cdot 2x\right)\right)^2 & \text{if } 0 \leq x < \frac{1}{2} \\ \left(\sin\left(\frac{\pi}{2}(2-2x)\right)\right)^2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$= \begin{cases} \left(\sin(\pi x)\right)^2 & \text{if } 0 \leq x < \frac{1}{2} \\ \left(\sin(\pi(1-x))\right)^2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$= \left(\sin(\pi x)\right)^2$$

Using symmetry of the function $x \mapsto \sin(\pi x)$



Now $f_4 \circ h(x) = f_4\left(\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\right)$

$$= 4\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\left(1-\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\right)$$

$$= 4\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\left(\cos\left(\frac{\pi x}{2}\right)\right)^2$$

$$= \left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right)^2$$

$$= \left(\sin(\pi x)\right)^2$$ by "double angle formula"

So $h(T(x)) = f_4(h(x))$ for all $x \in [0,1]$, so f_4 and T are topologically conjugate.