Queen Mary University of London

# Group Theory

## Week 6, Lecture 1, 2 & 3

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# Revision: Commutator subgroups

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# Homomorphisms

**Definition**

Suppose $G, H$ are groups.

- A **homomorphism** from $G$ to $H$ is a function $\phi : G \to H$ such that $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in G$.
- An **isomorphism** from $G$ to $H$ is a homomorphism which is also a bijection.
- $G, H$ are **isomorphic** (written $G \cong H$) if there is at least one isomorphism from $G$ to $H$.

# Homorphisms

**Examples**:

# Homorphisms

**Examples**:

# Homomorphisms

## Lemma

Suppose $G, H$ are groups and $\phi : G \to H$ is a homomorphism. Then $\phi(1) = 1$, and $\phi(g^{-1}) = (\phi(g))^{-1}$ for every $g \in G$.

## Exercise

If $\phi : G \to H$ is an isomorphism then $\phi^{-1} : H \to G$ is an isomorphism too.

# Homomorphisms

## Definition

Suppose $\phi : G \to H$ is a group homomorphism. The **image** of $\phi$ is the set $\mathrm{im}\phi = \left\{\phi(g) \mid g \in G\right\}$. The **kernel** of $\phi$ is the set $\ker\phi = \{g \in G : \phi(g) = 1\}$.

# Homomorphisms

**Examples**:

# Homomorphisms

**Lemma**

Suppose $\phi : G \to H$ is a group homomorphism. Then $\phi$ is injective if and only if $\ker \phi = \{1\}$.

# Homomorphisms

# Isomorphism Theorem

**First Isomorphism Theorem**

Suppose $G, H$ are groups and $\phi : G \to H$ is a homomorphism. Then

1. $im\phi \leq H$,
2. $ker\phi \trianglerighteq G$, and
3. $G/ker\phi \cong im\phi$.

**Proof**:

# Isomorphism Theorem

**proof**:

# First Isomorphism Theorem

**Example**: Det: $GL_n(F) \to F^\times$ induces an isomorphism $GL_n(F)/SL_n(F) \cong F^\times$

# Second Isomorphism Theorem

**Second Isomorphism Theorem**

Suppose $G$ is a group, $H \trianglerighteq G$, $K \trianglerighteq G$ and $K \subseteq H$. Then

1. $K \trianglerighteq H$,
2. $H/K \trianglerighteq G/K$, and
3. $(G/K)/(H/K) \cong G/H$.

# Second Isomorphism Theorem

**Proof**:

# Third Isomorphism Theorem

For the next isomorphism theorem, recall the product

$$HN = \{hn | h \in H,\ n \in N\}.$$

**Third Isomorphism Theorem**

Suppose $G$ is a group, $H \leq G$ and $N \trianglerighteq G$. Then

1. $N \trianglerighteq HN$,
2. $H \cap N \trianglerighteq H$, and
3. $H/(H \cap N) \cong (HN)/N$.

**Proof**:

# Third Isomorphism Theorem

**Proof**:

# Third Isomorphism Theorem

# Third Isomorphism Theorem

# Revision: Exams Style Questions

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# Exams Style Questions

**Question**: Let $\phi : G_1 \to G_2$ a homorphism.
(i) If $H_2 \trianglerighteq G_2$, then $\phi^{-1}(H_2) \trianglerighteq G_1$.
(ii) If $H_1 \trianglerighteq G_1$ and $\phi$ is an epimorphism then $\phi(H_1) \trianglerighteq G_2$.

**Proof** (i) If $x \in \phi^{-1}(H_2)$ and $a \in G_1$, then $\phi(x) \in H_2$ and so $\phi(axa^{-1}) = \phi(a)\phi(x)\phi(a)^{-1} \in H_2$ since $H_2$ is normal. We conclude $axa^{-1} \in \phi^{-1}(H_2)$.
(ii) Since $H_1$ is normal, we have $\phi(a)\phi(H_1)\phi(a)^{-1} \subseteq \phi(H_1)$. Since we assume $\phi$ is surjective, every $b \in G_2$ can be written as $b = \phi(a)$, $a \in G_1$. Therefore $b\phi(H_1)b^{-1} \in \phi(H_1)$.

**Remarks**: Note that with the choice $H_2 = \{e\}$ the theorem says that $ker\phi \trianglerighteq G_1$.

# QMplus Quiz

**Attempt Quiz 6 at QMplus page**

# Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order $n$.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c, \; ac = ca = b, \; bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and $n$ which are prime to $n$, with the group operation being multiplication modulo $n$.

# Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2 s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$