WEEK 6 NOTES

1. INHOMOGENEOUS WAVE EQUATIONS: WAVE EQUATIONS WITH A SOURCE

1.1. **Inhomogeneous wave equations on the real line.** We consider in this subsection the following initial value problem for inhomogeneous wave equation on the real line $\mathbb{R}$.

$$\begin{cases} U_{tt} - c^2 U_{xx} = \psi(x) \\ U(x,0) = f(x) \\ U_t(x,0) = g(x). \end{cases} \tag{1.1}$$

This is the mathematical model for the evolution of a vibrating string with a source of external force acting on it.

We will first apply the **Principle of Superposition** from the notes of Week 1.

Consider the following 2 equations:

$$\begin{cases} V_{tt} - c^2 V_{xx} = 0 \\ V(x,0) = f(x) \\ V_t(x,0) = g(x), \end{cases} \tag{1.2}$$

and

$$\begin{cases} W_{tt} - c^2 W_{xx} = \psi(x) \\ W(x,0) = 0 \\ W_t(x,0) = 0. \end{cases} \tag{1.3}$$

We observe that: if V is a solution to (1.2) and W is a solution to (1.3), then $U = V + W$ is a solution to (1.1).

By the D'Almbert's formula from Week 4, we get a solution V to (1.2) by

$$V(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

So we are left to solve (1.3), whose solution is given by the following Theorem called the Duhamel's Principle.

Theorem 1.1 (Duhamel's Principle on $\mathbb{R}$). *If $\tilde{W}$ is a solution to the equation*

$$\begin{cases} \tilde{W}_{tt} - c^2 \tilde{W}_{xx} = 0 \\ \tilde{W}(x,0) = 0 \\ \tilde{W}_t(x,0) = \psi(x), \end{cases} \tag{1.4}$$

then

$$W(x,t) = \int_0^t (\tilde{W}(x,t-s))ds \tag{1.5}$$

is a solution to (1.3)

Proof. We differentiate W with respect to t once and get

$$\begin{aligned}\frac{\partial}{\partial t}W(x,t) =&\frac{\partial}{\partial t}\int_0^t (\tilde{W}(x,t-s))ds\\ =&\tilde{W}(x,t-s)|_{s=t} + \int_0^t (\tilde{W}_t(x,t-s))ds\\ =&\tilde{W}(x,0) + \int_0^t (\tilde{W}_t(x,t-s))ds\\ =&\int_0^t (\tilde{W}_t(x,t-s))ds.\end{aligned}$$

Differentiating with respect to t again, we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}W(x,t) =&\frac{\partial}{\partial t}\int_0^t (\tilde{W}_t(x,t-s))ds\\ =&\tilde{W}_t(x,t-s)|_{s=t} + \int_0^t (\tilde{W}_{tt}(x,t-s))ds\\ =&\tilde{W}_t(x,0) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds\\ =&\psi(x) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds.\end{aligned}$$

Similarly, differentiating with respect to x twice, we get

$$\begin{aligned}\frac{\partial^2}{\partial x}W(x,t) =&\frac{\partial^2}{\partial x^2}\int_0^t (\tilde{W}(x,t-s))ds\\ =&\int_0^t (\tilde{W}_{xx}(x,t-s))ds.\end{aligned}$$

So combining these, we get

$$\begin{aligned}W_{tt} - c^2 W_{xx} =&\psi(x) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds - \int_0^t (\tilde{W}_{xx}(x,t-s))ds\\ =&\psi(x) + \int_0^t [\tilde{W}_{tt} - c^2\tilde{W}_{xx}](x,t-s)ds\\ =&\psi(x),\end{aligned}$$

where we used that $\tilde{W}$ satisfies $\tilde{W}_{tt} - c^2\tilde{W}_{xx} = 0$.

Moreover, at time $t = 0$, the initial values of W satisfy

$$\begin{aligned}W(x,0) =& \int_0^0 (\tilde{W}(x,t-s))ds = 0\\ W_t(x,0) =& \int_0^0 (\tilde{W}_t(x,t-s))ds = 0.\end{aligned}$$

□

On the other hand, we know the solution to (1.4) is given by D'Alembert's formula as follows

$$\tilde{W}(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(r)dr.$$

So, using the Duhamel's principle, we have

$$W(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds.$$

Combining with our earlier observation using the principle of superposition, the solution to (1.1) is then

(1.6)

$$\begin{aligned} U(x,t) =& V(x,t) + W(x,t) \\ =& \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds. \end{aligned}$$

Example 1.2. Solve the following inhomogeneous wave equation on the real line.

$$\begin{cases} U_{tt} - c^2 U_{xx} = \cos x \\ U(x,0) = -1 \\ U_t(x,0) = 1 \end{cases}$$

We can apply the formula (1.6) with $\psi(x) = \cos x, f(x) = -1, g(x) = 1$.

$$\begin{aligned} U(x,t) =& \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds \\ =& \frac{1}{2}[-1-1] + \frac{1}{2c}\int_{x-ct}^{x+ct} 1ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos r drds \\ =& -1 + \frac{2ct}{2c} + \frac{1}{2c}\int_0^t [\sin(x+ct-cs) - \sin(x-ct+cs)]ds \\ =& -1 + t + \frac{1}{2c}\frac{1}{-c}[-\cos(x+ct-cs)]|_0^t - \frac{1}{2c}\frac{1}{c}[-\cos(x-ct+cs)]|_0^t \\ =& -1 + t + \frac{\cos x}{c^2} - \frac{\cos(x+ct)+\cos(x-ct)}{2c^2} \end{aligned}$$

1.2. **Inhomogeneous equations on a finite interval.** Consider the following inhomogeneous wave equation for a vibrating string of finite length L and 2 fixed ends, a forcing term of ψ and zero initial position and velocity

$$\begin{cases} W_{tt} - c^2 W_{xx} = \psi(x), x \in [0,L] \\ W(0,t) = 0, W(L,t) = 0 \\ W(x,0) = 0, W_t(x,0) = 0. \end{cases} \tag{1.7}$$

Theorem 1.3 (Duhamel's Principle on interval)**.** *If* $\tilde{W}$ *is a solution to the equation*

$$\begin{cases} \tilde{W}_{tt} - c^2 \tilde{W}_{xx} = 0 \\ \tilde{W}(0,t) = 0, \tilde{W}(L,t) = 0 \\ \tilde{W}(x,0) = 0, \tilde{W}_t(x,0) = \psi(x), \end{cases} \tag{1.8}$$

then

$$W(x,t) = \int_0^t (\tilde{W}(x,t-s))ds \tag{1.9}$$

is a solution to (1.7)

The proof is similar to that of the proof of Theorem 1.1, we leave it as an exercise.

Using the principal of superposition and Theorem 1.3, we can again deduce as in the previous subsection the solutions to the inhomogeneous wave equations on the interval with arbitrary initial values.

The solutions to

$$\begin{cases} U_{tt} - c^2 U_{xx} = \psi(x), x \in [0, L] \\ U(0,t) = 0, U(L,t) = 0 \\ U(x,0) = f(x), U_t(x,0) = g(x). \end{cases} \tag{1.10}$$

Then $U(x,t) = V(x,t) + W(x,t)$ where V solves

$$\begin{cases} V_{tt} - c^2 V_{xx} = 0, x \in [0, L] \\ V(0,t) = 0, V(L,t) = 0 \\ V(x,0) = f(x), V_t(x,0) = g(x). \end{cases} \tag{1.11}$$

and W solve (1.7), both of which can be solved using the method of separation of variables in the last week.

Example 1.4. Solve the inhomogeneous equation on the interval

$$\begin{cases} U_{tt} - U_{xx} = -\sin(6x), x \in [0, 2\pi] \\ U(0,t) = 0, U(2\pi,t) = 0 \\ U(x,0) = 0, U_t(x,0) = 0 \end{cases} . \tag{1.12}$$

To solve the inhomogeneous equation, we first consider the following homogenous equation

$$\begin{cases} \tilde{W}_{tt} - \tilde{W}_{xx} = 0 \\ \tilde{W}(0,t) = 0, \tilde{W}(2\pi,t) = 0 \\ \tilde{W}(x,0) = 0, \tilde{W}_t(x,0) = -\sin(6x), \end{cases} \tag{1.13}$$

Using the theory of separation of variables from last week (equation (1.2) in Week 5 notes with $L = 2\pi$ and $c = 1$), we know that the solutions to this homogenous equation is

$$\begin{aligned} \tilde{W}(x,t) &= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{2\pi} \cos\frac{n\pi t}{2\pi} + b_n \sin\frac{n\pi x}{2\pi} \sin\frac{n\pi t}{2\pi} \\ &= \sum_{n=1}^{\infty} a_n \sin\frac{nx}{2} \cos\frac{nt}{2} + b_n \sin\frac{nx}{2} \sin\frac{nt}{2}, \end{aligned}$$

where

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} 0 \cdot \sin\frac{nx}{2} = 0, \forall n.$$

and

$$b_n = \frac{2}{n\pi} \int_0^{2\pi} [-\sin(6x)] \cdot \sin\frac{nx}{2} = \begin{cases} 0, n \neq 12 \\ \frac{2}{12\pi} \cdot (-\pi), n = 12 \end{cases}$$

So $a_n = 0$ for all n, $b_n = 0$ for all n except for $n = 12$, $b_{12} = -\frac{1}{6}$. Thus

$$\tilde{W}(x,t) = -\frac{1}{6} \sin(6x) \sin(6t).$$

By the Duhamel's principle on the interval Theorem 1.3, we then have solution to (1.12) is

$$\begin{aligned}U(x,t) =& \int_0^t \tilde{W}(x,t-s)ds\\=& \int_0^t -\frac{1}{6}\sin(6x)\sin[6(t-s)]ds\\=& \frac{1}{36}\sin(6x)\cos[6(t-s)]|_0^t\\=& -\frac{1}{36}\sin(6x) + \frac{1}{36}\sin(6x)\cos(6t).\end{aligned}$$

2. Basic ideas of elliptic equations

In this part of the course we will study the properties of elliptic equations in two dimensions (spatial). More precisely, we will look at the **Laplace equation**

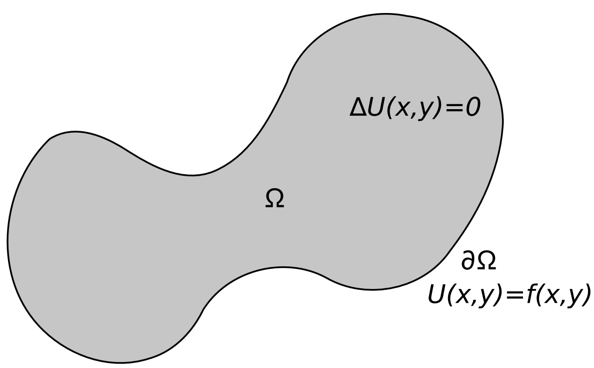
$$U_{xx} + U_{yy} = 0,$$

and the **Poisson equation**

$$U_{xx} + U_{yy} = f(x,y).$$

The Poisson equation is the inhomogeneous version of the Laplace equation.

Typically we will be interested in the so-called **Dirichlet problem** in which we solve the Laplace equation on a domain $\Omega \subset \mathbb{R}^2$ given that the value of U on the **boundary** $\partial\Omega$ of Ω is known.



Notation. *In what follows we write*

$$\Delta U = U_{xx} + U_{yy}.$$

The physicists notation is

$$\nabla^2 U = U_{xx} + U_{yy}.$$

The operator Δ (∇^2) is called the ***Laplacian***. *The reason for the physicists notation is that the Laplacian is the divergence of the gradient of a function $\Delta U = \nabla \cdot \nabla U$.*

The Laplace and Poisson equations arise from applications in physics (electrostatics, Newtonian gravity), fluid flows (steady state), soap films, elastic membranes, and also in pure mathematics (complex variables). As examples consider the wave equation in $1+2$ dimensions

$$U_{tt} = c^2(U_{xx} + U_{yy})$$

and the $1+2$ heat equation

$$U_t = \varkappa(U_{xx} + U_{yy}).$$

For both of these equations it is of interest to look for solutions which are independent of time —i.e. $U_t = 0$. These solutions describe the **asymptotic behaviour** —i.e. at late times. This is a statement that is hard to show and that is at the forefront of modern pde research.

2.0.1. *Harmonic functions.*

Definition 2.1. A function having second partial derivatives on a domain $\Omega \subset \mathbb{R}^2$ is called harmonic if $\Delta U = 0$ for all $(x, y) \in \Omega$.

Example 2.2.

(i) the function $U(x, y) = x + y$ is harmonic for all $\Omega \subset \mathbb{R}^2$;
(ii) similarly for the function $U(x, y) = x^2 - y^2$;
(iii) the function $U(x, y) = \ln(x^2 + y^2)$ for any domain Ω not containing the origin as the function $U(x, y)$ is not defined there.

2.0.2. *Relation to complex variables.* Let $f(z) = u(x, y) + \mathrm{i}v(x, y)$ be an analytic function with $z = x + \mathrm{i}y$. To verify that the function $f(z)$ is analytic on a domain Ω one can make use of the *Cauchy-Riemann* equations:

$$v_y = u_x, \tag{2.1a}$$

$$v_x = -u_y. \tag{2.1b}$$

Applying $\partial/\partial y$ to equation (2.1a) one has that

$$v_{yy} = u_{xy} = -v_{xx}$$

where the second equality follows from (2.1b). Thus, one has that

$$v_{xx} + v_{yy} = 0,$$

that is, the imaginary part of analytic function is harmonic. A similar relation follows for the real part u.

Note. *This observation indicates a very deep connection between pde's and complex variables!*

3. Laplace equation in Polar coordinates

Before studying the general properties of the Laplace and Poisson equations, let us consider some explicit solutions using separation of variables.

3.1. Separation of variables in polar coordinates.

The method of separation of variables can be used to find solutions to the Laplace equation in settings with circular symmetry —i.e. a disk or an annulus.

Given the polar coordinates (r, θ) given by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

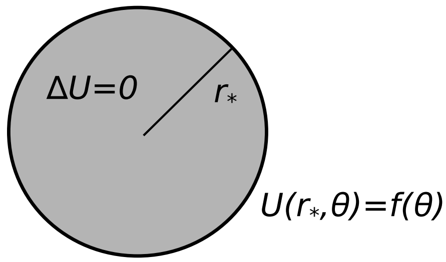
the Laplacian can be expressed as

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Consider the boundary value problem for the Laplace equation in which the value of the solution is given on a circumference of a disk of radius r_* —namely,

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0,$$
$$U(r_*, \theta) = f(\theta),$$

see the figure below:



Our task is to find the solution $U(r,\theta)$ in the interior of the circumference (**disk**). Following the general strategy of the method of separation of variables we look for solutions of the form

$$U(r,\theta) = R(r)\Theta(\theta).$$

Plugging into the Laplace equation in polar coordinates one obtains the expression

$$\Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing the above expression by $R\Theta/r^2$ and rearranging one finds that

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}.$$

The left hand side of the above expression depends only on r while the right hand side only on θ. Thus, each must be equal to some separation constant k —namely:

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = k, \qquad -\frac{\Theta''}{\Theta} = k,$$

or

$$r^2 R'' + rR' - kR = 0,$$
$$\Theta'' + k\Theta = 0.$$

The Θ-equation. This equation is used to set the value of k. Observe that we need periodic solutions so $k > 0$. In the following we write $k = m^2$. Then

$$\Theta(\theta) = A\cos m\theta + B\sin m\theta.$$

To enforce periodicity we require that

$$U(r,\theta) = U(r,\theta + 2\pi),$$
$$U_\theta(r,\theta) = U_\theta(r,\theta + 2\pi).$$

Observing that

$$\cos m(\theta + 2\pi) = \cos(m\theta + 2\pi m) = \cos m\theta$$

if $m \in \mathbb{N}$ (and similarly for $\sin m\theta$) then $m \in \mathbb{N}$.

The R-equation. Following the previous discussion one has that the equation for $R(r)$ takes the form
$$r^2R'' + rR' - m^2R = 0.$$
We look for solutions to this equations of the form
$$R(r) = r^\alpha,$$
for some constant α. It follows then that
$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - m^2r^\alpha = 0$$
so that
$$(\alpha^2 - m^2)r^\alpha = 0.$$
Hence, $\alpha^2 = m^2$ —that is,
$$\alpha = \pm m.$$
So the general solution for the R equation is
$$R(r) = C_m r^m + \frac{D_m}{r^m}.$$
For $m = 0$ one needs to do more work as there must be two independent solutions. In that case one has the equation
$$r^2R'' + rR' = 0.$$
If $r \neq 0$ the latter implies
$$\begin{aligned} r\frac{dR'}{dr} + R' &= 0 \\ [rR']' &= 0. \end{aligned}$$
Integrating both sides give
$$\begin{aligned} rR'(r) &= D_0 \\ R'(r) &= \frac{D_0}{r}, \end{aligned}$$
from where a further integration gives
$$R(r) = C_0 + D_0 \ln r.$$

The general solution. Combining the whole of the previous discussion one finds that the general solution to the Laplace equation in polar coordinates is given by

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty} \Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta). \tag{3.1}$$