Queen Mary
University of London

BC-BB-FC (Campus M)

Group Theory

Week 5, Lecture 1, 2 & 3

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Assessment 1 — week 8

Deadline (Friday week-8)

Week 1 - Week 5
topics included in Mid-term.

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Conjugate subgroups

Revision: Conjugacy. $f, g \in G$ $f \sim_G g$

if $\exists\, k \in G$ $g = kfk^{-1}$

Note: If $H \leqslant G$, $f, g \in H$

$$f \sim_H g \Rightarrow f \sim_G g$$ however the converse is not true.

$$\not\Leftarrow$$

Example: $g = (2\,3\,4)$ & $h = (1\,3\,4)$, Are $g \sim_{S_4} h$ in S_4.

Yes, bcs, they have same cycle type.

Conjugate subgroups

Are they conjugate in S_4.

Answer is No.

Definition

Suppose G is a group, $H \leq G$ and $g \in G$. Define

$$gHg^{-1} = \{ghg^{-1} | h \in H\}$$

$kHk^{-1} = \{khk^{-1} : h \in H\}$ for some $k \in G$.

Conjugate Subgroup of G.

$$S_4 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \dots \right\}$$

$$|S_4| = 4 \times 3 \times 2 = 24$$

Conjugate subgroups

Example: $G = S_4$.

$$H = \langle \underline{\overset{g}{(1\,2\,3\,4)}} \rangle$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}^{\checkmark}$$

$$g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\,3)(2\,4)$$

$$g^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}^{\checkmark} = (1\,4\,3\,2)$$

$$g^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$H = \{ id, (1234), (13)(24), (1432) \}$$

$$k = (123)$$

$$kHk^{-1} = \{ id, (2314), (21)(34), (2413) \}$$

$$L = \langle (2314) \rangle$$ Verify this

Conjugate subgroups

Lemma

Suppose G is a group, $H \leq G$ and $g \in G$. Then gHg^{-1} is a subgroup of G.

Subgroup Test

Proof.

$$1 = g \cdot 1 \cdot g^{-1} \in gH \cdot g^{-1} \Rightarrow gHg^{-1} \neq \phi$$

$$gh_1g^{-1},\ gh_2g^{-1} \in gHg^{-1}$$

$$(gh_1g^{-1}) \cdot (gh_2g^{-1})^{-1} = gh_1g^{-1} \cdot gh_2^{-1} \cdot g^{-1}$$

$$= g\underbrace{h_1h_2^{-1}}_{\substack{\cap \\ H}}g^{-1} \in gHg^{-1}$$

Subgroup of G.

Conjugate subgroups

Definition: Suppose G is a group, and $H, K \leq G$. We say that H and K are **conjugate** if $K = gHg^{-1}$ for some $g \in G$.

Conjugacy is an equivalence relation on the set of subgroups of G.

Example: (1) $G = \mathcal{S}_4$.

In S_4 $\langle (1\,2\,3) \rangle = \{ id, (123), (321) \}$

is conjugate $\langle (124) \rangle = \{ id, (124), (421) \}$

2) $\langle \underline{f} \rangle$ will always be conjugate to $\langle \underline{hfh^{-1}} \rangle$ for any h & f in G.

Conjugate subgroups

Example: $\mathcal{D}_8$: $\langle s \rangle = \{1, s\}$ is conjugate to $\langle r^2, s \rangle = \{1, r^2 s\}$

Example:

$G = S_5$ $\qquad |S_5| = 120$

$$\left.\begin{array}{l} H_1 = \{h \in S_5 : h(1) = 1\} \\ H_3 = \{h \in S_5 : h(3) = 3\} \end{array}\right\} \quad 4 \times 3 \times 2 = 24$$

are conjugate to each other

$r = (13)$ ✓

$r H_1 r^{-1} = H_3$

$\underbrace{(13)\, h\, (13)}(3) =$

$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} (3)$

$\big(h(13)(3)\big) = \underline{1}$

Normal Subgroups and Quotient Groups

$(13)(1) = 3$ ✓

Definition

Suppose G is a group and $\underline{N \leq G}$. We say N is **normal** in G if

$$\underline{gng^{-1} \in N} \qquad \text{for all } n \in N \text{ and } g \in G.$$

it is conjugate to itself in G.

We write $N \trianglerighteq G$ to mean that N is a normal subgroup of G. We use the symbols $\trianglerighteq, \trianglelefteq, \triangleright, \triangleleft$ in the obvious way.

Note that if $N \trianglerighteq G$, then $gNg^{-1} = N$ for every $g \in G$. In other words, the only subgroup of G conjugate to N is N itself.

Examples:

i) $G \trianglerighteq G$

ii) $\{1\} \trianglerighteq G$

iii) $Z(G) \trianglerighteq G$

$n \in Z(G)$

$h \cdot n \cdot h^{-1} = n \in Z(G)$

$h \in G$

iv) Any Subgroup of Abelian group is Normal.

Normal Subgroups and Quotient Groups

Example: $\mathcal{D}_8 = \{1, s, r, r^2, r^3, \underline{rs}, r^2 s, r^3 s\}$

$\boxed{s \cdot r = r^{-1} s}$

$N = \{1, \underline{s}\}$

$M = \{1, r, \overset{\checkmark}{r^2}, \overset{\checkmark}{r^3}\}$

$\overset{\checkmark}{r} \cdot s \cdot \overset{\checkmark}{r^{-1}} = r \cdot s \cdot \underline{r^{-1} \cdot s} \cdot s^{-1}$

$= \underline{r \cdot s} \cdot \underline{s \cdot r}\, s^{-1} = r \cdot r \cdot s^{-1} = r^2 s \notin N$

$\underline{s \cdot r} \cdot s^{-1} = r^{-1} \cdot s \cdot s^{-1} = r^{-1} = r^3$

$(rs) \cdot r^2 \cdot (rs)^{-1} = r s \underline{\cdot r} \cdot r \cdot s^{-1} \cdot r^{-1}$

$= \underline{r \cdot r^{-1}} \cdot s\, \underline{r}\, s^{-1} r^{-1}$

$= r^{-1} \cdot \underline{s \cdot s^{-1}} \cdot r^{-1} = r^{-2} = r^2$

M Normal Subgroup $\mathcal{D}_8$

Normal Subgroups and Quotient Groups

ccl = conjugacy class

$\text{ccl}(e) = \{e\}$

$\text{ccl}(x) = \{gxg^{-1} : x \in G\}$

Lemma

A subgroup is normal if and only if it is a union of conjugacy classes.

Example: $S_4 = \{id, (1234), (12)(34), \cdots \}$

$V = \{id, (12)(34), (13)(24), \underline{(14)(23)}\}$

$V \leq S_4$

$f = (12)(34) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$g = (13)(24) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

Normal Subgroups and Quotient Groups

identity element is always conjugate to itself.

$$ccl((12)(34)) = \{(13)(24), (14)(23), (12)(34)\}$$

$$V = ccl\{e\} \cup ccl\{(12)(34)\}$$

Hence Normal

Normal Subgroups and Quotient Groups

Lemma

Suppose G is a group and $N \leq G$, and that $|G : N| = 2$. Then $N \trianglerighteq G$.

Order of N in $G = 2$ ✓

Proof 2 Left cosets

$1 \cdot N = N$, & $G \setminus N$

$gN \neq N$

2 Right cosets

$N \cdot 1$ & $G \setminus N$

$N \cdot g \neq N$

There are only Two right coset, N & $G \setminus N$

Take $g \in G$ & $n \in N$.

Suppose $g \notin N \Rightarrow Ng$ is one right coset $Ng \neq N$

also $gn \notin N \Rightarrow Ngn \neq N$

$\Rightarrow Ngn = Ng \Rightarrow gng^{-1} \in N$

Normal Subgroups and Quotient Groups

N is normal.

Lemma v.v. Imp

Suppose G is a group and $N \leq G$ is normal if and only if the right cosets coincide with the left cosets.

$\Longrightarrow$

$\Longleftarrow$

Proof: $N \trianglelefteq G$

$$gN \ni gn = gn \cdot g^{-1} \cdot g \in Ng \Rightarrow gN \subseteq Ng$$

$$Ng \ni ng = gg^{-1} \cdot ng = gn_1 \in gN$$

This means $Ng \subseteq gN$

$$gN = Ng$$

Normal Subgroups and Quotient Groups

Conversely, Suppose that left cosets are equal to right coset for any $g \in G$

$gN = Ng$. ✓

for any $n \in N$ ✓

$gn = n_1 g$ for some $n_1 \in N$

$\Rightarrow \underline{gng^{-1} = n_1} \in N$

Normal Subgroup.

Normal Subgroups and Quotient Groups

v.v.imp

Definition

Suppose G is a group and $N \trianglerighteq G$. The **quotient group** G/N is the set of all cosets of N, with group operation

$$(Ng_1)\cdot(Ng_2) = Ng_1g_2$$

$$\underline{(Ng)(Nh) = Ngh.}$$

$$Ng,\ Nh \in G/N$$

Claim: The group operation $\trianglerighteq$ is well defined.

$$\tilde{g}_1 \in Ng_1,\quad \tilde{g}_2 \in Ng_2 \quad \Rightarrow \quad \tilde{g}_1\tilde{g}_2 \in Ng_1g_2$$

$$\tilde{g}_1 = n_1g_1,\quad \tilde{g}_2 = n_2g_2,\quad n_1, n_2 \in N$$

$$\tilde{g}_1 \cdot \tilde{g}_2 = n_1 \cdot \underline{g_1 \cdot n_2} \cdot g_2 = n_1(\underbrace{g_1 n_2 g_1^{-1}}_{\tilde{n}_2 \in N} \cdot g_1)\cdot g_2$$

$$= n_1 \tilde{n}_2 \cdot g_1 \cdot g_2 \in Ng_1 \cdot g_2$$

Normal Subgroups and Quotient Groups

Proposition

Suppose G is a group and $N \trianglerighteq G$. The **quotient** G/N is a group with operation

$$(Ng)(Nh) = Ngh. \qquad \text{v.v. Imp}$$

Proof: Associativity $(Ng_1)(Ng_2 \cdot Ng_3)$

$$= (Ng_1)\,(Ng_2 \cdot g_3) = Ng_1 g_2 \cdot g_3 \text{ — ①}$$

$$(Ng_1 \cdot Ng_2)(Ng_3) = (Ng_1 \cdot g_2)(Ng_3)$$

$$= Ng_1 g_2 g_3 \text{ — ②}$$

Associative $(N1)(Ng) = N1 \cdot g = Ng$

Normal Subgroups and Quotient Groups

$N \cdot 1 = N$ identity

Lemma

Suppose G is a finite group and $N \trianglerighteq G$. Then $|G/N| = |G|/|N|$.

Proof: By definition $|G/N| = |G:N|$ & by Lagrange's Theorem

$$\frac{|G|}{|N|} = |G:N|$$

For Inverse

for $g \in G$, $\exists\, g^{-1} \in G$

& so is $Ng^{-1} \in G/N$

$$(Ng)^{-1} = (Ng^{-1})$$

$$Ng^{-1} \circ Ng = Ng \cdot g^{-1} = N$$

$$Ng \circ Ng^{-1} = Ng^{-1} \cdot g = N$$

Normal Subgroups and Quotient Groups

Campus M : FF - BP - BZ

Revision:

Normal Subgroups: $N \trianglelefteq G$

- $\forall\, h \in G \quad hNh^{-1} = \{hnh^{-1} : n \in N\}$

  Subgroup of G $= N$

- $\forall\, h \in G \quad hN = Nh$
- G/N form a group.

Normal Subgroups and Quotient Groups

Examples: i) $A_n \trianglelefteq S_n \Leftarrow |S_n : A_n| = 2$

ii) $C_n \trianglelefteq D_{2n} \Leftarrow |D_{2n} : C_n| = 2$

all the rotation

Products of groups

Definition

Suppose G is a group and $\overset{\checkmark}{H}, \overset{\checkmark}{N} \leq G$. Define

$$HN = \{hn \mid h \in H,\ n \in N\}.$$

You might hope that if H and N are subgroups of G then so is HN. But in fact this is not the case.

Example: $G = S_3$

$$A = \{id, (12)\} \leq S_3$$

$$B = \{id, (23)\} \leq S_3$$

$$AB = \{id, \underline{(12)}, \underline{(23)}, \underline{(12)(23)} = \underline{(123)}\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

~~A~~

Products of groups

Proposition

Suppose G is a group, $H \leq G$ and $N \trianglerighteq G$. Then $HN \leq G$. If in addition $H \trianglerighteq G$, then $HN \trianglerighteq G$.

proof: 1) $H \leq G$, $N \trianglelefteq G$, HN

$1 \in HN$, $1 = 1 \cdot 1 \in HN$

let $h_1 n_1 \in HN$, $h_2 n_2 \in HN$

$$(h_1 n_1) \cdot (h_2 n_2)^{-1} = \underbrace{h_1 h_2^{-1}}_{H} \underbrace{h_2 \cdot n_1 \cdot n_2^{-1} h_2^{-1}}_{N}$$

$\in HN$

2) $H, N \trianglelefteq G$

$g(hn)g^{-1}$

$$\underbrace{ghg^{-1}}_{H} \underbrace{gng^{-1}}_{N}$$

$\in HN$

Products of groups

Definition

Suppose G, H are groups. The **direct product** $G \times H$ is the set

$$G \times H = \{(g, h) | g \in G,\ h \in H\}$$

with group operation

$$(g, h)(g', h') = (gg', hh').$$

Examples: $G = D_8$

$H = \{1, \underline{rs}\}$

$N = \{1, \underline{r^2}\}$

$s \cdot r = \underline{r^{-1}} s$

$$r \cdot s \cdot r^2 = r \cdot \underline{s \cdot r} \cdot r = \underline{r r^{-1}} \cdot s \cdot r = r^{-1} s$$

$$HN = \{1, \underline{rs}, r^2, rs \cdot \underline{r^2} = r \cdot r^{-1} \cdot s \cdot r = \underline{r^{-1} s} = \underline{r^3 s}\}$$

Products of groups

Example

$$r\underline{s\cdot r}s = r\cdot r^{-1}s\cdot s = s^2 = 1$$

$$rs\cdot r^2 = r^3 s$$

$$\left.\begin{aligned}
r^3 s\cdot r^3 s &= r^3 s\cdot r\cdot r^2 s \\
&= r^3\ r^{-1}s\cdot r^2 s \\
&= r^2\ s\cdot r\cdot r\cdot s \\
&= r^2 r^{-1}s\ \cdot r\cdot s \\
&= r\cdot s\cdot rs \\
&= r r^{-1}s\cdot s = 1
\end{aligned}\right\} \leq D_8$$

Centralizer subgroups

Definition

G is a group and $A \subseteq G$

$$C_G(A) = \{g \in G : \forall a, \; gag^{-1} = a\}$$

$$\Longleftrightarrow ga = ag$$

$$C_G(A) = \{g \in G \mid \underline{\forall a \in A} : gag^{-1} = a\}$$

Claim: $C_G(A) \leq G$.

Proof: $1 \in C_G(A)$

$$g \cdot 1 \cdot g^{-1} = 1$$

$$g \cdot 1 = 1 \cdot g$$

$$g_1, g_2 \in C_G(A) \Rightarrow \forall a \quad g_1 a = a g_1$$

Centralizer subgroups

$$g_2 a = a g_2$$

$$g_1 g_2^{-1} \in C_G(A) \iff g_1 g_2^{-1} a = a g_1 g_2^{-1}$$

or

$$g_1 \underbrace{g_2^{-1} a g_2}_{} g_1^{-1} = g_1 a g_1^{-1} = a$$

$$\implies g_1 g_2^{-1} \in C_G(A).$$

Commutator subgroups

$$[f,g] = \underline{f \cdot g}\,\underline{f^{-1} g^{-1}} \cdot$$

Definition

Suppose G is a group and $f, g \in G$. The **commutator** of f and g (written $[f,g]$) is the element $fgf^{-1}g^{-1}$. The **commutator subgroup** $\underline{G'}$ (also called the **derived subgroup**) is the subgroup of G generated by all the commutators in G.

$$G' \leq G$$

Proposition

Suppose G is a group. Then $\underline{G' \trianglerighteq G}$.

proof $hnh^{-1} \in G'$, $h \in G$, $n \in G'$

1) $[fg]^{-1} = (f \cdot g \cdot f^{-1} g^{-1})^{-1}$

$= g f g^{-1} f^{-1} = [g f]$

$h[fg]h^{-1} = h f \cdot g \cdot f^{-1} \cdot g^{-1} \cdot h^{-1}$

$[f_1, g_1], [f_2, g_2]$

$[f_3, g_3], [\underline{f_3, g_3}]^{-1}$

$(f_3 g_3 f_3^{-1} g_3^{-1})^{-1}$

. . - - -

$[\quad\quad]$

Commutator subgroups

$$= \underline{h \cdot f \cdot h^{-1}} \cdot \underline{h \cdot g \cdot h^{-1}} \cdot \underline{h \cdot f^{-1} \cdot h^{-1}} \cdot \underline{h g^{-1} \cdot h^{-1}}$$

$$= [hfh^{-1}, hgh^{-1}]$$

Proposition

Suppose G is a group and $\underline{N \trianglerighteq G}$. Then G/N is abelian ✓ if and only if $\underline{G' \subseteq N}$.

Proof: G/N is abelian $\Longleftrightarrow (N f)(N g)$ $\quad f, g \in G$

$$= (Ng)(Nf)$$

$$\Longleftrightarrow Nfg = Ngf$$

by coset Lemma ✓ $\Longleftrightarrow fgf^{-1}g^{-1} \in N$

$[f,g] \in N$

Commutator subgroups

$G' \subseteq N$

Examples

$$D_{10} = \{1, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\}$$

$$C_{D_{10}}\{r^3\} = \{\forall g \in D_8 \mid g \cdot r^3 = r^3 \cdot g\}$$

$$= \{\forall g \in D8 \mid g r^3 \cdot g^{-1} = r^3\}$$

$$r^2 \cdot r^3 \cdot r^{-2} = r^{5-2} = r^3 \quad = \{1, r, r^2, r^3, r^4\} = C_{D_{10}}(r^3)$$

Exams Style Questions

Normal Subgroup

Tutorial

Lemma

Let G be a finite group and let $H \leq G$. Show that for every $g \in G$, the set $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G. Now explain briefly why the following result holds: if H is the only subgroup of G with cardinality $|H|$, then H must be normal in G.

Solution: Fix $g \in G$. To show that gHg^{-1} is a subgroup of G it's sufficient to show that it is nonempty, and that for every $f_1, f_2 \in gHg^{-1}$ we have $f_1 f_2^{-1} \in gHg^{-1}$. To see the former we just note that $1 = g1g^{-1}$. To see the latter, let $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$. We have $(gh_2g^{-1})^{-1} = gh_2^{-1}g^{-1}$ so $gh_1g^{-1}(gh_2g^{-1})^{-1} \in gHg^{-1}$, using the fact that $h_1h_2^{-1}$. If H is the unique subgroup of G with cardinality $|H|$, then for every $g \in G, gHg^{-1}$ is a subgroup of G with cardinality $|H|$ and therefore must equal H. This implies that H is normal by the definition of normality. (students should use the correct reasoning along these lines: the equation $|H| = |gHg^{-1}|$ need not be proved.)

$$gh_1g^{-1}(gh_2g^{-1})^{-1} = gh_1g^{-1}(gh_2^{-1}g^{-1})$$

Exams Style Questions

$$= g h h_2^{-1} g^{-1} \in gHg^{-1}$$

Question:

1. Consider the element r^3 of the dihedral group $\mathcal{D}_{10}$. Find the *centraliser* of r^3 in $\mathcal{D}_{10}$.
   Solution: All rotations commute with r^3 so the centraliser contains all five rotations (including the identity). It follows by Lagrange's theorem that the centraliser either consists only of rotations, or consists of all elements of $\mathcal{D}_{10}$. Since $r^3s = sr^{-3} = sr^2 \neq sr^3$, r^3 does not commute with s, so the centraliser is not the whole of $\mathcal{D}_{10}$ and therefore must be $\{1, r, r^2, r^3, r^4\}$.
2. Now instead consider the element r^3 of the dihedral group $\mathcal{D}_{12}$. Find the *centraliser* of r^3 in $\mathcal{D}_{12}$.
   Solution: All rotations commute with r^3, so the centraliser has cardinality at least six. Since in this group $r^3s = sr^{-3} = sr^3$, r^3 commutes with s and therefore commutes with all elements of $\mathcal{D}_{12}$.
3. Write down the *centre* of the group $\mathcal{D}_{10}$.
   Solution: All rotations (except the identity) fail to commute with s, so the centraliser is just $\{1\}$.

$$D_{12} = \{1, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$$

QMplus Quiz

$C_{D_{12}}\{r^3\} = \{ \qquad \}$

In D_{10}

$$s \cdot r^3 s^{-1} = \underline{s \cdot r} \cdot r^2 \cdot s^{-1} = r^{-1} \cdot \underline{s \cdot r} \cdot r \cdot s^{-1}$$

$$= r^{-1} \cdot r^{-1} \cdot \underline{s \cdot r} \cdot s^{-1}$$

Attempt Quiz 5 at QMplus page

$$= r^{-2} \cdot r^{-1} \cdot \underline{s \cdot s^{-1}}$$

$$= r^{-3} = r^2$$

$$s \cdot r^3 \cdot s^{-1} = r^2 \neq r^3$$

$\Rightarrow s$ Not in the Centre of r^3

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c, \ ac = ca = b, \ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

$$sr \cdot r^3 \cdot (sr)^{-1} = r^2$$

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

  $$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

  $$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$