1. The eigenvalue problem is

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0 \\ X(L) = 0 \end{cases}$$

The general solution is

$$\cancel{X_n(x)=} \quad X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}X)$$

$$X'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$X'(0) = C_2\sqrt{\lambda} = 0$$

implies $C_2 = 0$

$C_1 \neq 0$,

using $X(L) = 0$, we get

$$\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot L) = 0$$

$$\sqrt{\lambda}L = \frac{\pi}{2} + n\pi.$$

So $$X_n = \frac{(\frac{1}{2}+n)^2\pi^2}{L^2}$$

and $X_n(x) = \cos\left[(\frac{1}{2}+n)\frac{\pi x}{L}\right]$

Selected solutions to problem set 5.

2. We first find solutions using separation of variables For solutions of the form

$$u(x,t) = X(x)\,T(t)$$

$$X\ddot{T} = c^2 X'' \cdot T.$$

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda.$$

$$\begin{cases} \ddot{T} = -c^2\lambda T \\ X'' = -\lambda X \end{cases}$$

Now we solve X using the condition

$$u_x(0,t) = u_x(\pi,t) = 0,$$

we get

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0 \\ X'(\pi) = 0. \end{cases} \qquad (*1)$$

claim: $\lambda \geq 0$. (This is similar to the argument in lecture notes)

Proof of claim: Multiply both sides by X, get

$$XX'' = -\lambda X^2.$$

$$XX'' + \lambda X^2 = 0$$

So $0 = \int_0^\pi (XX'' + \lambda X^2)\,dx \underset{\text{integration by parts}}{=} XX'\Big|_0^\pi - \int_0^\pi (X')^2 dx + \lambda \int_0^\pi X^2 dx$

(3)

Since $XX'\Big|_0^\pi = 0$ by the condition, we have

$$0 = -\int_0^\pi (X')^2 dx + \lambda \int_0^\pi X^2 dx$$

$$\int_0^\pi (X')^2 dx = \lambda \cdot \int_0^\pi X^2 dx,$$

$\Rightarrow \lambda \geq 0.$

so (*1) has solutions (using the method for solving 2nd order ODEs) given by

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

Differentiate get, $X'(x) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x).$

By the boundary conditions, we get

$$X'(0) = C_2 \sqrt{\lambda} \cos(0) = 0$$

so $C_2 = 0$,

and $C_1 \neq 0$.

$$X'(\pi) = -\sqrt{\lambda} \sin(\sqrt{\lambda} \pi) = 0$$

This implies $\sqrt{\lambda} \pi = n\pi$

$\lambda_n = n^2$ are eigenvalues

and $X_n(x) = \cos(nx)$, $n = 0, 1, \cdots$

To solve T, we have

$$\ddot{T} = -\lambda_n c^2 T = -n^2 c^2 T,$$

when $n=0$, $T_0(t) = d_1 + d_2 t$

when $n \geq 1$, $T_n(t) = d_1\cos(nct) + d_2 \sin(nct)$

So
$$\begin{cases} u_n(x,t) = a_n \cos(nx)\cos(nct) + b_n \cos(nx)\sin(nct), & n \neq 0 \\ u_0(x,t) = a_0 + b_0 t, & n = 0 \end{cases}$$

And
$$u(x,t) = (a_0 + b_0 t) + \sum_{n=1}^{\infty} \left[a_n \cos(nx)\cos(nct) + b_n \cos(nx)\sin(nct)\right]$$

Now, use $u(x,0) = 0$, we get

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0,$$

so $a_0 = 0$, $a_n = 0$

So $u(x,t) = b_0 t + \sum_{n=1}^{\infty} b_n \cos(nx) + \sin(nct)$

~~Use $u_t(x,0) = 0$, we get~~

So ~~Since~~ $u_t(x,t) = b_0 + \sum_{n=1}^{\infty} (nc)\, b_n \cos(nx)\cos(nct)$

Thus $u_t(x,0) = b_0 + \sum_{n=1}^{\infty} (nc)\, b_n \cos nx$,

Using $u_t(x,0)$ ~~$=0$, we then get~~
$= \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

we then get $b_0 = \frac{1}{2}$, $b_2 = \frac{1}{4c}$, $b_n = 0$ for $n \neq 0, 2$

So the solution is

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\cos(2x)\sin(2ct).$$

3. The Fourier coefficients are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} |x|\,dx$$

$$= \frac{1}{L}\int_0^L x\,dx$$

$$= \frac{1}{L}\cdot\frac{L^2}{2} = \frac{L}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} |x|\cdot\cos\left(\frac{\pi n x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L x\cos\frac{\pi n x}{L}\,dx$$

$$= \frac{2}{L}\cdot\frac{x\cdot L}{\pi n}\sin\frac{\pi n x}{L}\Big|_0^L - \frac{2}{L}\int_0^L \frac{L}{\pi n}\sin\frac{\pi n x}{L}dx$$

$$= \frac{-2}{L}\cdot\frac{-L^2}{(\pi n)^2}\cos\left(\frac{\pi n x}{L}\right)\Big|_0^L$$

$$= \frac{2L}{(\pi n)^2}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{2L}{(\pi n)^2}\left[(-1)^n - 1\right].$$

and thus $\hat{f}(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{(\pi n)^2}\left[(-1)^n - 1\right]\cos\left(\frac{\pi n x}{L}\right)$

Solution to PS 5 Q6 :

Consider for $x \in [0, L]$ the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad , \; x \in [0, L] \\ u(0,t) = 0 \; , \; u_x(L,t) = 0 \quad \leftarrow \text{Mixed boundary conditions} \\ u(x,0) = x \, , \; u_t(x,0) = 0 \quad \leftarrow \text{Initial conditions} \end{cases}$$

Find the solution by separation of variables.

Step 1: As we did in the lecture notes, we first consider solutions of the form $u(x,t) = X(x)\,T(t)$.

The equation becomes $X \cdot \ddot{T} - c^2 X'' T = 0$

(upper dot is t derivative and "prime" is x derivative)

So
$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$
is independent of both x and t.

Thus
$$\frac{\ddot{T}}{cT} = \frac{X''}{X} = -\lambda$$
is a constant.

This give 2 ODES

$$\begin{cases} X'' + \lambda X = 0 & \text{(a)} \\ \ddot{T} + c^2 \lambda T = 0 & \text{(b)} \end{cases}$$

Using the boundary conditions

$u(0,t)=0, u_x(L,t)=0$, we get

$$X(0)=0, \quad X'(L)=0$$

Combining with (a), get an eigenvalue problem

$$\begin{cases} X''+\lambda X=0 \\ X(0)=0, \ X'(L)=0 \end{cases} \qquad (*)$$

Claim: $\lambda > 0$.

proof of claim: Multiply (a) by X and integrate, get

$$\int_0^L X \cdot X'' + \lambda \int X^2 = 0$$

$$X \cdot X' \Big|_0^L - \int (X')^2 + \lambda \int X^2 = 0$$

Using the boundary conditions, we have

$$X \cdot X' \Big|_0^L = X(L)X'(L) - X(0)X'(0) = 0$$

Since $X \neq 0$ is nontrivial, we have $\lambda > 0$. #

Knowing $\lambda > 0$, the general solution to $(*)$ is

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

The first boundary conditions read

$$0 = C_1 \cdot \cos 0 + C_2 \sin 0 = C_1$$

C_1, C_2 cannot be both zero because X is non-trivial.

so $C_2 \neq 0$.

The second boundary condition is then

$$0 = X'(L) = C_2 \cdot \sqrt{\lambda} \cos(\sqrt{\lambda} x) \Big|_{x=L} = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$$

this implies $\sqrt{\lambda} L = \frac{\pi}{2} + n\pi$, $n = 1, 2, \cdots$

The eigenvalues are thus $\lambda_n = \dfrac{(\frac{1}{2}+n)^2 \pi^2}{L^2}$

The eigenfunctions are $X_n(x) = \sin(\sqrt{\lambda_n} x)$

$$= \sin \frac{(\frac{1}{2}+n)\pi x}{L}$$

Knowing λ_n, we solve (b) and get

$$T_n(t) = a_n \cos \frac{(\frac{1}{2}+n)\pi c t}{L} + b_n \sin \frac{(\frac{1}{2}+n)\pi c t}{L}.$$

The general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} a_n \sin \frac{(\frac{1}{2}+n)\pi x}{L} \cos \frac{(\frac{1}{2}+n)\pi c t}{L}$$

$$+ \sum_{n=1}^{\infty} b_n \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+n)\pi c t}{L}$$

<u>Step 2:</u> Next, we use the initial values to determine the a_n's and b_n's.

Differentiate the general solution with respect to t, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} - \frac{a_n \cdot (\frac{1}{2}+n)\pi c}{L} \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+n)\pi c t}{L}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n \cdot (\frac{1}{2}+n)\pi c}{L} \sin \frac{(\frac{1}{2}+n)\pi x}{L} \cos \frac{(\frac{1}{2}+n)\pi c t}{L}$$

The initial conditions then read (plugging in $t=0$)

$$\begin{cases} x = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{(\frac{1}{2}+n)\pi x}{L} & (c) \\ 0 = u_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \frac{(\frac{1}{2}+n)\pi c}{L} \cdot \sin \frac{(\frac{1}{2}+n)\pi x}{L} & (d) \end{cases}$$

we get $b_n = 0$ for all n.

multiply the equation (c) by $\sin \frac{(\frac{1}{2}+m)\pi x}{L}$ and integrate from 0 to L, get

$$\int_0^L x \cdot \sin \frac{(\frac{1}{2}+m)\pi x}{L} = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+m)\pi x}{L}. \quad (e)$$

Recall $\int_0^L \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+n)\pi x}{L} = \begin{cases} \frac{L}{2}, & n=m \\ 0, & n \neq m \end{cases}$

(e) then gives

$$a_m = \left[\int_0^L x \cdot \sin \frac{(\frac{1}{2}+m)\pi x}{L}\right] \cdot \frac{2}{L}$$

$$= \left[\frac{-x \cdot L \cdot \cos \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)^2 \pi}\Bigg|_0^L + \int_0^L \frac{L \cdot \cos \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)\pi} dx\right] \cdot \frac{2}{L}$$

$$= \left[\frac{-L^2 \cos(\frac{1}{2}+m)\pi}{(\frac{1}{2}+m)^2 \pi} + 0 + \frac{L^2 \sin \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)^2 \pi^2}\Bigg|_0^L\right] \cdot \frac{2}{L}$$

$$= \left[0 + 0 + \frac{L^2 \sin\left(\frac{1}{2}+m\right)\pi}{\left(\frac{1}{2}+m\right)^2 \pi^2} - 0 \right] \cdot \frac{2}{L}$$

$$= \frac{8L \cdot (-1)^n}{(1+2m)^2 \pi}$$

And so

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8L (-1)^n}{\pi^2 \cdot (1+2n)^2} \sin\left[\left(\frac{1}{2}+n\right)\frac{\pi x}{L}\right] \cos\left[\left(\frac{1}{2}+n\right)\frac{\pi c t}{L}\right]$$

PS 5 Q 2:

First consider solutions with separated variables
$$u(x,t) = X(x)T(t).$$
plug into the equation gives
$$X \cdot \ddot{T} - c^2 X'' T = 0$$
$$\frac{\ddot{T}}{cT^2} = \frac{X''}{X} = -\lambda \quad (A)$$

The 2nd idenity in (A) together with the boundary conditions gives the eigenvalue problem
$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \ X'(\pi) = 0 \end{cases}$$
The general solutions for X are
$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x).$$
It's derivative is
$$X'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$
The first bondary condition gives
$$0 = X'(0) = C_2 \sqrt{\lambda},$$
This implies $C_2 = 0$ and $C_1 \neq 0$.

The second boundary condition then gives

$$0 = X'(\pi) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)$$

Thus $\sqrt{\lambda}\pi = n\pi$, $n = 0, 1, 2, \cdots$

So eigenvalues are $\lambda_n = n^2$, $n = 0, 1, 2 \cdots$

eigenfunctions are $X_n(x) = \cos(nx)$

For $n \geq 1$, knowing λ_n, we can solve

$$\frac{\ddot{T}}{c^2 T^2} = T \quad \text{and get}$$

$$T_n(t) = a_n \cos(cnt) + b_n \sin(cnt)$$

For $n=0$, i.e. $\lambda = 0$, we also have

we have $X_0(x) = \cos 0 = 1$

and solving $T_0'' = 0$ gives $T_0 = a_0 + b_0 t$.

So the general solutions is given by

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= a_0 + b_0 t + \sum_{n=1}^{\infty} a_n \cos(nx)\cos(cnt) + \sum_{n=1}^{\infty} b_n \cos(nx)\sin(cnt)$$

Its time derivative is

$$u_t(x,t) = b_0 - \sum_{n=1}^{\infty} c \cdot n \cdot a_n \cos(nx)\sin(cnt) + \sum_{n=1}^{\infty} c \cdot n \cdot b_n \cos(nx)\cos(cnt)$$

plug in $t = 0$, we get

$$0 = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\frac{1}{2} + \frac{\cos 2x}{2} = u_t(x,0) = b_0 + \sum_{n=1}^{\infty} c \cdot n \cdot b_n \cos(nx)$$

Thus we can determine the cofficients.

using that $\sin mx$ $\cos nx$ are independent if $m \neq n$, we get

$a_n = 0$ for all n

$b_n = 0$ for all n except for $n = 0$ or 2

and $b_0 = \frac{1}{2}$, $c \cdot 2 \cdot b_2 = \frac{1}{2} \Rightarrow b_2 = \frac{1}{4c}$

so the solution to this Question is

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\cos(2x)\sin(2ct)$$

PS 5 Q4:

The Fourier series is (as defined in Week 6 notes)

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{\pi}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{\pi}\right)$$

with
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$
$$= \frac{1}{\pi}\cdot(-\cos x)\Big|_0^{\pi}$$
$$= \frac{2}{\pi}$$

and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{\pi} dx$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos nx \, dx$

$= \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx$

We compute $\int_0^\pi \sin x \cos nx \, dx = -\cos x \cos nx \Big|_0^\pi - \int_0^\pi (-\cos x)(-n \sin nx) dx$

$= (-1)^n + 1 - n \int_0^\pi \cos x \sin nx \, dx$

$= (-1)^n + 1 - n \sin x \sin nx \Big|_0^\pi + n^2 \int_0^\pi \sin x \cos nx \, dx$

$= 1 + (-1)^n + n^2 \int_0^\pi \sin x \cos nx \, dx$

Thus $\int_0^\pi \sin x \cos nx \, dx = \frac{-1 - (-1)^n}{n^2 - 1}$

and $a_n = \frac{2[-1-(-1)^n]}{\pi \cdot (n^2-1)}$

Noticing $f(x) = |x|$ is even, so all $b_n = 0$, $n = 1, 2, \cdots$

Thus $\hat{f}(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2[-1-(-1)^n]}{\pi \cdot (n^2-1)} \cos(nx)$