Logistic family of maps

Definition The logistic family is the family of functions f_μ defined by

$$f_\mu(x) = \mu x(1-x)$$

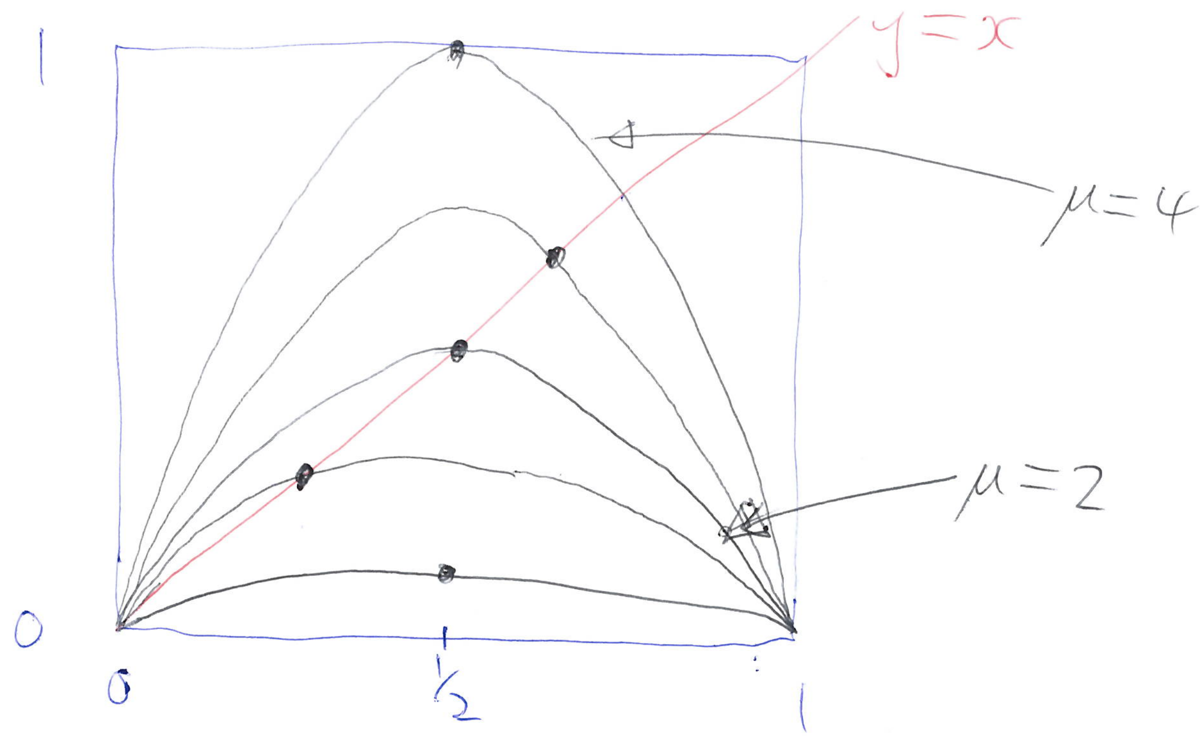
where $\mu > 0$ is a parameter

As usual, we will study the dynamical system given by f_μ

i.e. $x_{n+1} = \mu x_n(1-x_n)$

- First studied by R. May (1976) to model insect population dynamics
- Today we treat this system as a model which has a transition to chaos

By convention we shall assume $x \in [0,1]$ (i.e. we consider $f_\mu : [0,1] \to [0,1]$, and $\mu \in [0,4]$)



Graphs of f_μ for various parameters μ

Derivative : $f_\mu'(x) = \mu - 2\mu x$

The critical point (a maximum) is at $x = \frac{1}{2}$ for all values of parameter μ.

The maximum value of f_μ is then $f_\mu(\frac{1}{2}) = \frac{\mu}{4}$

Note $f_\mu(x) = f_\mu(1-x)$ for all x, so the graph of f_μ is symmetric about the point $1/2$.

Fixed points

$$f_\mu(x) = x$$

i.e. $\mu x(1-x) = x$

i.e. $0 = \mu x^2 + (1-\mu)x$

i.e. $x = 0$ or $0 = \mu x + (1-\mu)$

i.e. $x = \dfrac{\mu-1}{\mu} = 1 - 1/\mu$

So $x = 0$ and $x = \dfrac{\mu-1}{\mu}$ are the fixed points of f_μ, though note that if $\mu < 1$ then the 'fixed point' $\dfrac{\mu-1}{\mu}$ is negative therefore outside of $[0,1]$, so we do not consider it (since we are thinking of f_μ as a map $[0,1] \circlearrowleft$).

When are these fixed points attracting?

We first calculate $|f_\mu'(x)|$:

$$|f_\mu'(x)| = |\mu - 2\mu x|$$

$$= \begin{cases} |\mu| & \text{if } x = 0 \\ |2-\mu| & \text{if } x = \frac{\mu-1}{\mu} \end{cases}$$

So for each fixed point we can (using a previous Theorem) say:

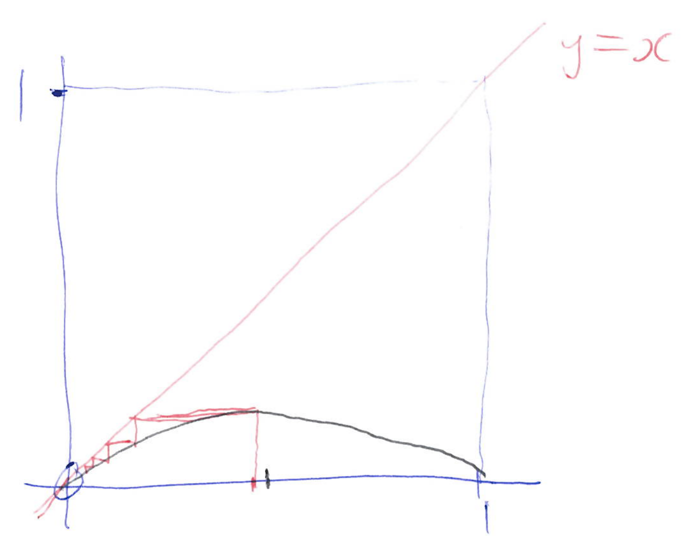
- The fixed point 0 is attracting if $0 \leq \mu < 1$, and repelling if $1 < \mu \leq 4$
- The fixed point at $\frac{\mu-1}{\mu}$ is attracting if $|2-\mu| < 1$

  ie. if $1 < \mu < 3$,

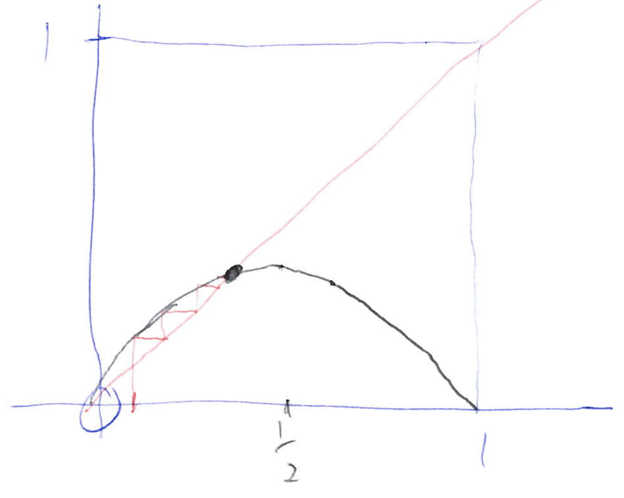
  and is repelling if $3 < \mu \leq 4$.

Remark So $\mu = 1$ is a key transition parameter, since the fixed point 0 stops being attracting, and the 'new' fixed point $\frac{\mu - 1}{\mu}$ is 'born'.

Graph of f_μ (for $0 \leq \mu < 1$)

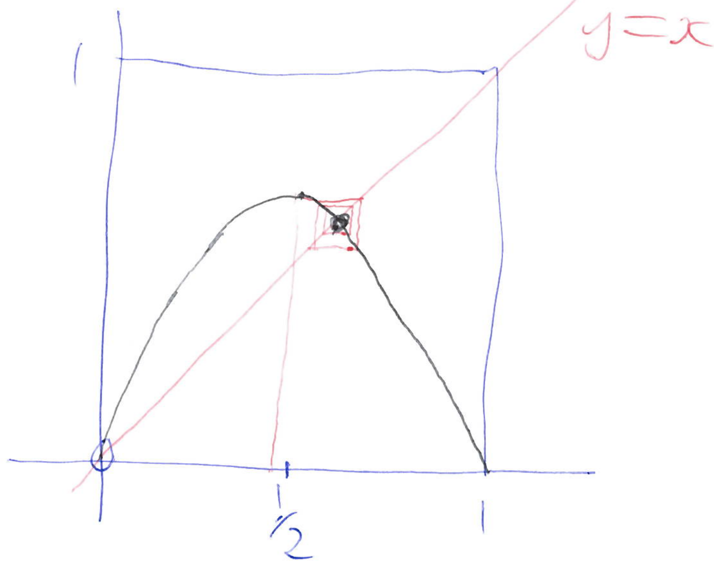


Graph of f_μ (for $1 < \mu < 2$)

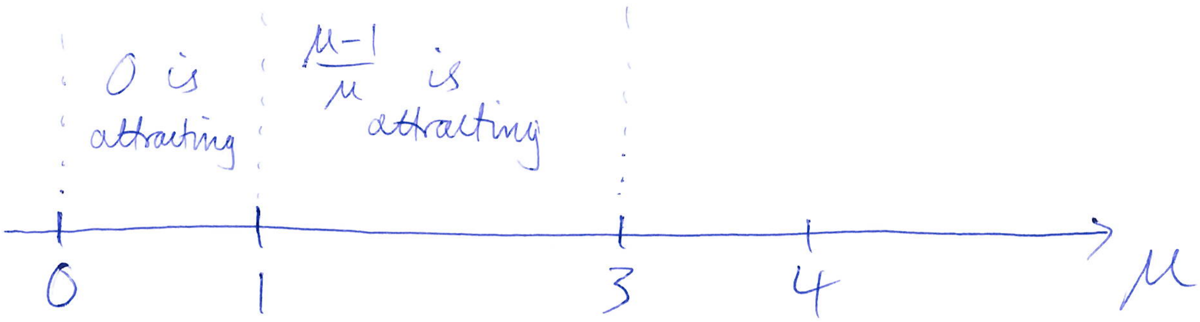


Graph of f_μ

(for $2 < \mu < 3$)



So far we can summarise as follows:



For $\mu > 3$, both fixed points 0 and $\frac{\mu-1}{\mu}$ are <u>repelling</u>, but what is happening dynamically?

We will see that $\mu = 3$ marks the 'birth' of a period-2 orbit, and for those $\mu > 3$ which "are not too much larger than 3" this period-2 orbit is <u>attracting</u>.

Let's investigate period-2 points in detail. Period-2 points are solutions to the equation $f_\mu^2(x) = x$

ie. $$f_\mu^2(x) - x = 0$$

ie. $$f_\mu(\mu x(1-x)) - x = 0$$

ie. $$\mu(\mu x(1-x))(1 - \mu x(1-x)) - x = 0$$

ie. $$-x + (\mu^2 x - \mu^2 x^2)(1 - \mu x + \mu x^2) = 0$$

check this

ie. $$x(-\mu x + \mu - 1)(\mu^2 x^2 - (\mu^2+\mu)x + (\mu+1)) = 0$$

Note this is $f_\mu(x) - x$
ie. solutions are fixed points

The points of least period 2 will be roots of this quadratic

The points of least period 2 are the roots of $\mu^2 x^2 - (\mu^2+\mu)x + \mu + 1$

i.e. The ~~p~~ two points, p_+ and p_-, of least period 2 are

$$p_\pm = \frac{1}{2\mu^2}\left(\mu^2+\mu \pm \sqrt{(\mu^2+\mu)^2 - 4\mu^2(\mu+1)}\right)$$

(just using the "quadratic formula")

$$= \frac{1}{2\mu}\left(\mu+1 \pm \sqrt{(\mu+1)(\mu-3)}\right)$$

Notice that if $\mu = 3$ then both p_+ and p_- are equal to the fixed point $\frac{\mu-1}{\mu} = \frac{2}{3} = \frac{1}{6}(3+1 \pm \sqrt{0})$

$$= \frac{1}{2\mu}\left(\mu+1 \pm \sqrt{0}\right)$$

If $\mu < 3$ then $p_\pm$ do not exist (we do not consider non-real solutions).

Is the 2-cycle $\{p_+, p_-\}$ attracting? Is it repelling?

More precisely, for which parameter values μ is the 2-cycle attracting or repelling?

We will analyse the modulus of the multiplier M of the 2-cycle, where

$$
\begin{aligned}
M &= \left| (f_\mu^2)'(p_+) \right| \\
&= \left| f_\mu'(f_\mu(p_+)) \cdot f_\mu'(p_+) \right| \\
&= \left| f_\mu'(p_-) \cdot f_\mu'(p_+) \right| \\
&= \left| (\mu - 2\mu p_-) \cdot (\mu - 2\mu p_+) \right|
\end{aligned}
$$

$$= \left| \mu^2 - 2\mu^2(p_- + p_+) + 4\mu^2 p_- p_+ \right|$$

check

$$= \left| \mu^2 - 2\mu(\mu+1) + 4(\mu+1) \right|$$

$$= \left| 4 + 2\mu - \mu^2 \right|$$

Recall that the 2-cycle $\{p_+, p_-\}$ is attracting if $M < 1$ and repelling if $M > 1$

Now $M = 1$ when

<u>either</u> ① $4 + 2\mu - \mu^2 = +1$

<u>or</u> ② $4 + 2\mu - \mu^2 = -1$

Case ①: $4 + 2\mu - \mu^2 = 1 \iff 0 = \mu^2 - 2\mu - 3$

$\iff 0 = (\mu - 3)(\mu + 1)$

So $M = 1$ when $\mu = 3$ (we ignore the case $\mu = -1$ since we only consider $\mu \in [0, 4]$)

Case 2 : $4 + 2\mu - \mu^2 = -1$

$$\Longleftrightarrow \quad \mu^2 - 2\mu - 5 = 0$$

$$\Longleftrightarrow \quad \mu = \frac{1}{2}\left(2 \pm \sqrt{4 + 20}\right)$$

$$= \frac{1}{2}\left(2 \pm 2\sqrt{6}\right)$$

$$= 1 \pm \sqrt{6}$$

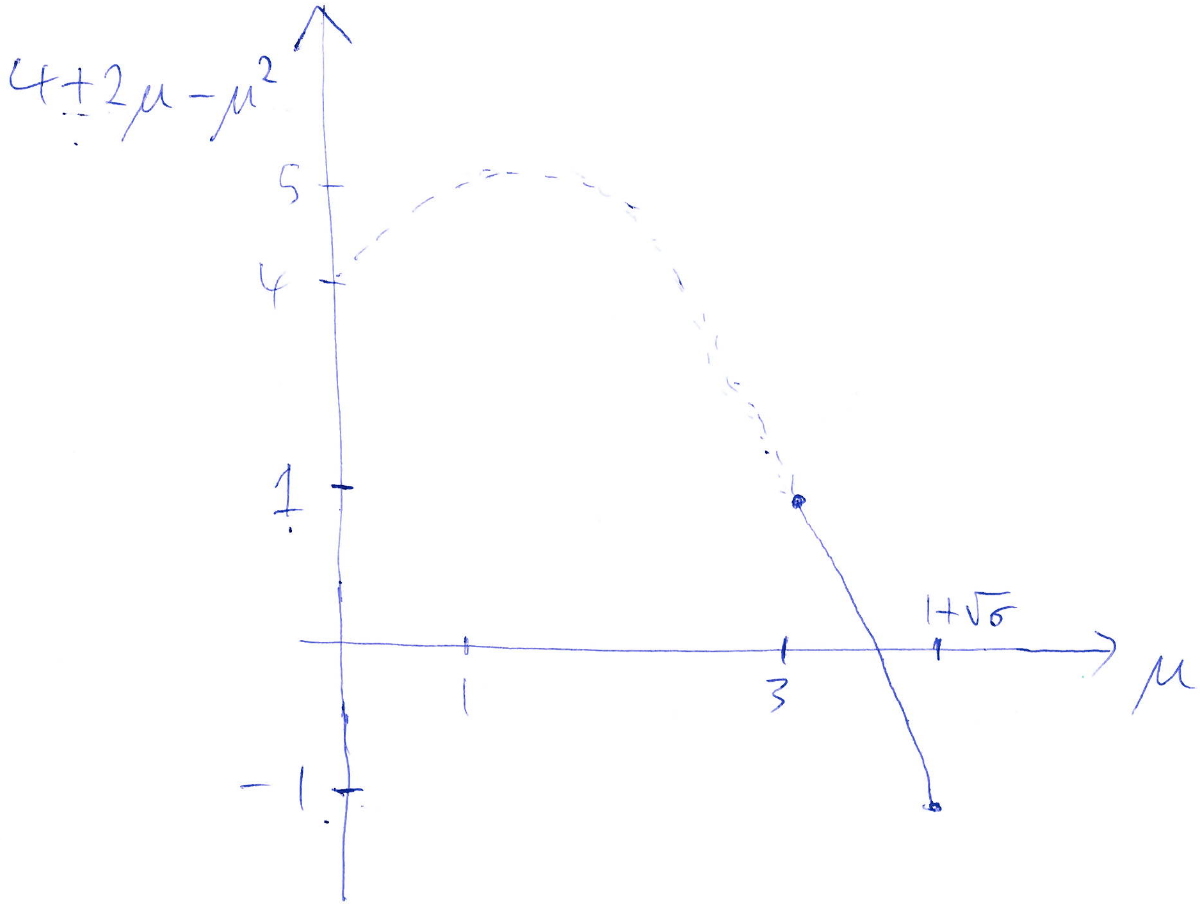
So $M = 1$ when $\mu = 1 + \sqrt{6}$

$\simeq 3.449\ldots$

(we ignore the value $1 - \sqrt{6}$, since it is negative)

So $M < 1$ for $3 < \mu < 1 + \sqrt{6}$

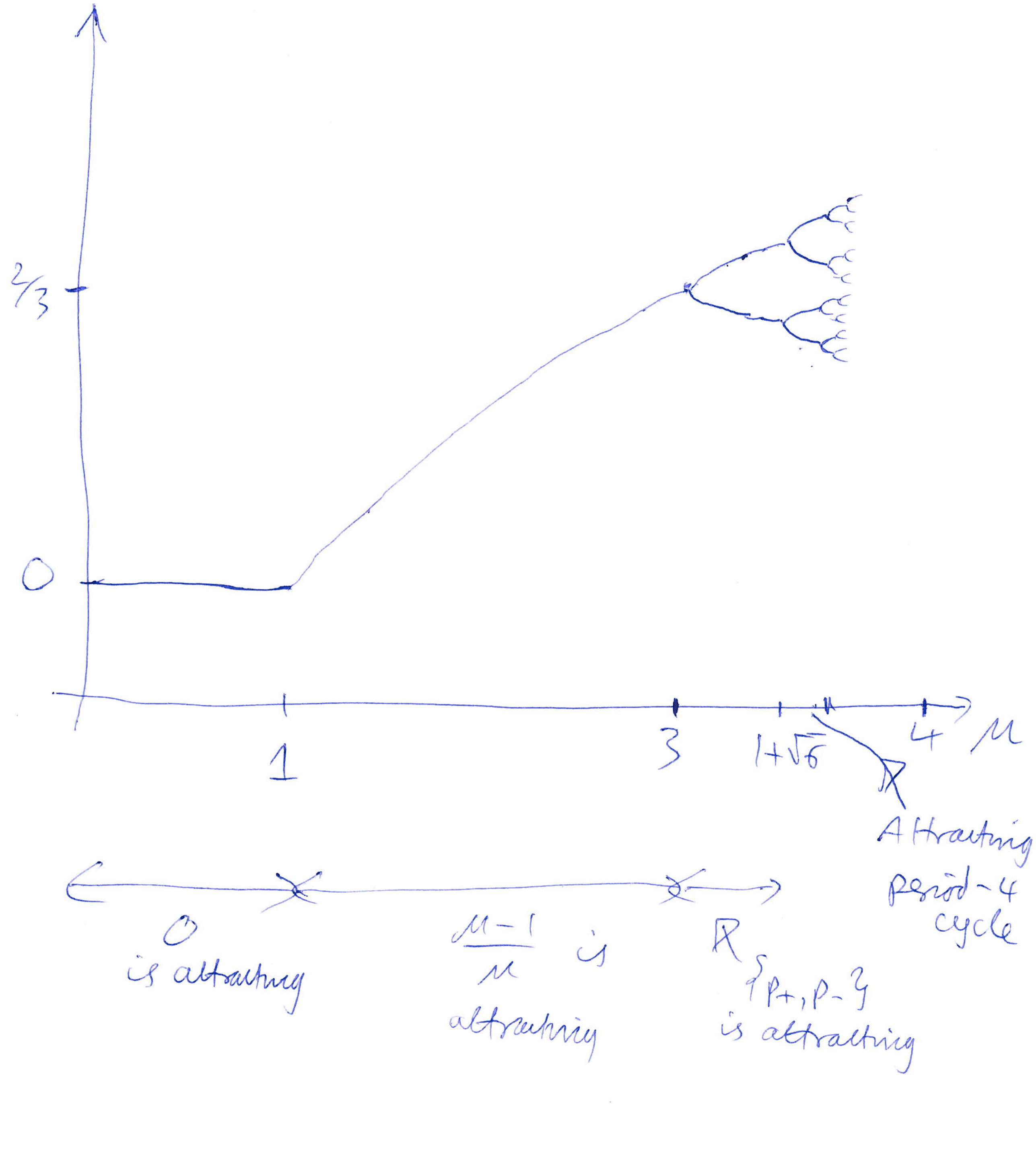
i.e. The 2-cycle $\{p_+, p_-\}$ is attracting if $3 < \mu < 1 + \sqrt{6}$



$4 \pm 2\mu - \mu^2$
5
4
1
-1
1
3
$1+\sqrt{6}$
μ

The evolution of the attracting periodic orbit, as μ increases, can be summarised by the following so-called

<u>bifurcation diagram</u>

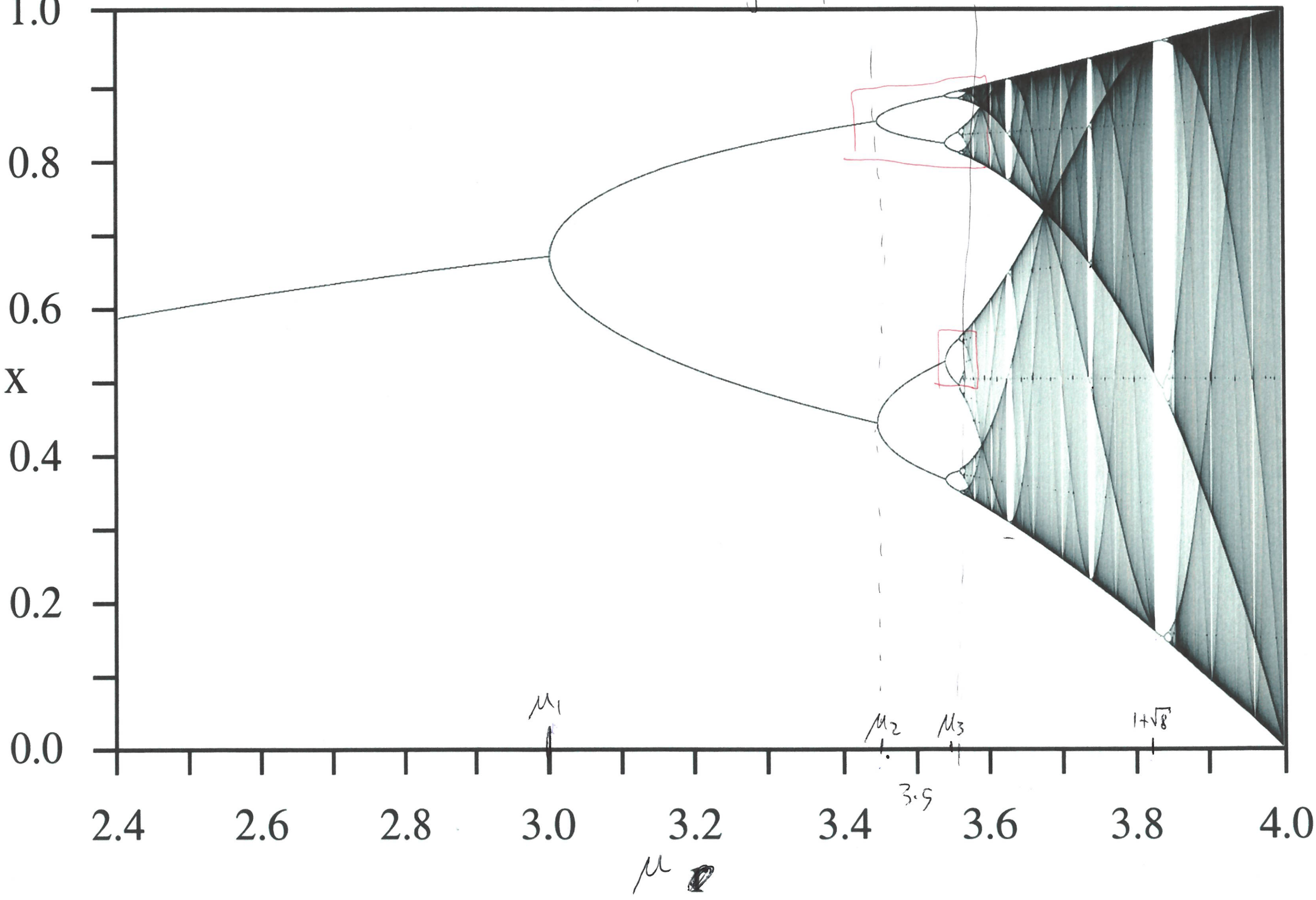


It turns out that at each stage, the loss of ~~attracting~~ 'attractingness'/attraction of the period-2^n orbit gives rise to an attracting periodic orbit of least period 2^{n+1}.

This is called a *period-doubling bifurcation*.

In fact, since we have many such period-doubling bifurcations, this is sometimes referred to as a *period-doubling cascade*.

Bifurcation Diagram



If we let μ_n denote the special values of μ at which these period-doubling bifurcations occur, then:

$$(\mu_0 = 1)$$

$$\mu_1 = 3$$

$$\mu_2 = 1+\sqrt{6} \simeq 3.449$$

$$\mu_3 = 3.544\ldots$$

$$\mu_4 = 3.564\ldots$$

$$\mu_5 = 3.568\ldots$$

$$\vdots$$

$$\mu_\infty = 3.569946\ldots = \lim_{n\to\infty} \mu_n$$

The sequence $(\mu_n)_{n=0}^{\infty}$ converges to some value μ_∞, as $n \to \infty$

On (μ_0, μ_1), $\frac{\mu-1}{\mu}$ is attracting

On (μ_1, μ_2), a 2-cycle is attracting

On (μ_2, μ_3), a 4-cycle is attracting

$\vdots$

On (μ_n, μ_{n+1}), a 2^n-cycle is attracting

The lengths of these bifurcation parameter intervals are decreasing rapidly (in fact, at some geometric rate, i.e. exponentially fast)

We can write

$$d_k = \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}$$

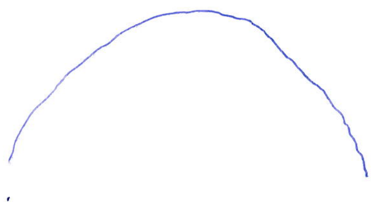
It was observed experimentally by

Feigenbaum (and others) around 1975, that the sequence $(d_k)_{k=1}^{\infty}$ has a limit

$$d_\infty \simeq 4.669202\ldots$$

This is called the *Feigenbaum constant* or the *Feigenbaum ratio*.

It was also observed that we get the *same* "universal value" d_∞ for *all* parametrised families of maps *looking like* the logistic family



e.g. $g_\lambda(x) = \lambda \sin(\pi x)$

A natural question is: what happens for $\mu > \mu_\infty$?

All period-2^n orbits are repelling beyond the value μ_∞.

For some parameters μ there are other attracting periodic orbits whose period is <u>not</u> a power of 2.

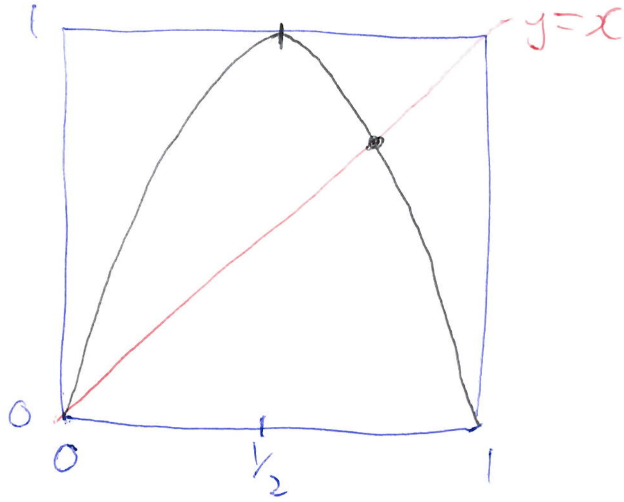
For some parameters μ, the behavior of f_μ is "chaotic".

<u>Example</u> $\mu = 4$: $\underbrace{f_4(x)}_{f(x)} = 4x(1-x)$

Fixed points: $x = 0$ and $x = 3/4$ $\left(= \frac{\mu - 1}{\mu}\right)$

Now $f'(x) = 4 - 8x$

So $|f'(x)| = \begin{cases} 4 & \text{if } x = 0 \\ |-2| = 2 & \text{if } x = 3/4 \end{cases}$



So both fixed points are repelling.

Period-2 points: $f^2(x) = x$

$$\iff f(f(x)) = x$$

$$\iff f(4x(1-x)) = x$$

$$\iff 4 \cdot 4x(1-x)\big(1 - 4x(1-x)\big) = x$$

$$\iff -x(4x-3)(16x^2 - 20x + 5) = 0$$

So the roots of this quadratic

$p_\pm = \dfrac{5 \pm \sqrt{5}}{8}$ constitute a 2-cycle

This 2-cycle is <u>repelling</u>, because

$|(f^2)'(p_+)| = |f'(p_+) \cdot f'(p_-)| = \ldots = |-4|$

$= 4 > 1$