Group Theory

Week 5, Lecture 1, 2 & 3

Dr Lubna Shaheen

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Conjugate subgroups

Revision:

Conjugate subgroups

Definition

Suppose G is a group, $H \leq G$ and $g \in G$. Define

$$gHg^{-1} = \{ghg^{-1} | h \in H\}$$

Conjugate subgroups

Example: $G = S_4$.

Conjugate subgroups

Lemma

Suppose G is a group, $H \leq G$ and $g \in G$. Then gHg^{-1} is a subgroup of G.

Conjugate subgroups

Definition: Suppose G is a group, and $H, K \leq G$. We say that H and K are **conjugate** if $K = gHg^{-1}$ for some $g \in G$.

Conjugacy is an equivalence relation on the set of subgroups of G.

Example: $G = \mathcal{S}_4$.

Conjugate subgroups

Example: $\mathcal{D}_8$

Normal Subgroups and Quotient Groups

Definition

Suppose G is a group and $N \leq G$. We say N is **normal** in G if

$$gng^{-1} \in N \qquad \text{for all } n \in N \text{ and } g \in G.$$

We write $N \trianglerighteq G$ to mean that N is a normal subgroup of G. We use the symbols $\trianglerighteq, \trianglelefteq, \triangleright, \triangleleft$ in the obvious way.

Note that if $N \trianglerighteq G$, then $gNg^{-1} = N$ for every $g \in G$. In other words, the only subgroup of G conjugate to N is N itself.

Normal Subgroups and Quotient Groups

Example:

Normal Subgroups and Quotient Groups

Lemma

A subgroup is normal if and only if it is a union of conjugacy classes.

Normal Subgroups and Quotient Groups

Normal Subgroups and Quotient Groups

Lemma

Suppose G is a group and $N \leq G$, and that $|G : N| = 2$. Then $N \trianglerighteq G$.

Normal Subgroups and Quotient Groups

Lemma

Suppose G is a group and $N \leq G$ is normal if and only if the right cosets coincide with the left cosets.

Normal Subgroups and Quotient Groups

Normal Subgroups and Quotient Groups

Definition

Suppose G is a group and $N \trianglerighteq G$. The **quotient group** G/N is the set of all cosets of N, with group operation

$$(Ng)(Nh) = Ngh.$$

Claim: The group operation $\trianglerighteq$ is well defined.

Normal Subgroups and Quotient Groups

Proposition

Suppose G is a group and $N \trianglerighteq G$. The **quotient** G/N is a group with operation

$$(Ng)(Nh) = Ngh.$$

Proof:

Normal Subgroups and Quotient Groups

Lemma

Suppose G is a finite group and $N \trianglerighteq G$. Then $|G/N| = |G|/|N|$.

Normal Subgroups and Quotient Groups

Normal Subgroups and Quotient Groups

Products of groups

Definition

Suppose G is a group and $H, N \leq G$. Define

$$HN = \left\{ hn \middle| \, h \in H, \; n \in N \right\}.$$

You might hope that if H and N are subgroups of G then so is HN. But in fact this is not the case.

Products of groups

Proposition

Suppose G is a group, $H \leq G$ and $N \trianglerighteq G$. Then $HN \leq G$. If in addition $H \trianglerighteq G$, then $HN \trianglerighteq G$.

Products of groups

Definition

Suppose G, H are groups. The **direct product** $G \times H$ is the set

$$\{(g,h) | g \in G,\ h \in H\}$$

with group operation

$$(g,h)(g',h') = (gg',hh').$$

Products of groups

Example

Centralizer subgroups

Definition

G is a group and $A \subseteq G$

$$C_G(A) = \{g \in G \mid \forall a \in A : gag^{-1} = a\}$$

Claim: $C_G(A) \leq G$.

Centralizer subgroups

Commutator subgroups

Definition

Suppose G is a group and $f, g \in G$. The **commutator** of f and g (written $[f, g]$) is the element $fgf^{-1}g^{-1}$. The **commutator subgroup** G' (also called the **derived subgroup**) is the subgroup of G generated by all the commutators in G.

Proposition

Suppose G is a group. Then $G' \trianglerighteq G$.

Commutator subgroups

Proposition

Suppose G is a group and $N \trianglerighteq G$. Then G/N is abelian if and only if $G' \subseteq N$.

Commutator subgroups

Examples

Exams Style Questions

Normal Subgroup

Lemma

Let G be a finite group and let $H \leq G$. Show that for every $g \in G$, the set $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G. Now explain briefly why the following result holds: if H is the only subgroup of G with cardinality $|H|$, then H must be normal in G.

Solution: Fix $g \in G$. To show that gHg^{-1} is a subgroup of G it's sufficient to show that it is nonempty, and that for every $f_1, f_2 \in gHg^{-1}$ we have $f_1 f_2^{-1} \in gHg^{-1}$. To see the former we just note that $1 = g1g^{-1}$. To see the latter, let $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$. We have $(gh_2g^{-1})^{-1} = gh_2^{-1}g^{-1}$ so $gh_1g^{-1}(gh_2g^{-1})^{-1} \in gHg^{-1}$, using the fact that $h_1 h_2^{-1}$. If H is the unique subgroup of G with cardinality $|H|$, then for every $g \in G, gHg^{-1}$ is a subgroup of G with cardinality $|H|$ and therefore must equal H. This implies that H is normal by the definition of normality. (students should use the correct reasoning along these lines: the equation $|H| = |gHg^{-1}|$ need not be proved.)

Exams Style Questions

Question:

1. Consider the element r^3 of the dihedral group $\mathcal{D}_{10}$. Find the *centraliser* of r^3 in $\mathcal{D}_{10}$.
   Solution: All rotations commute with r^3 so the centraliser contains all five rotations (including the identity). It follows by Lagrange's theorem that the centraliser either consists only of rotations, or consists of all elements of $\mathcal{D}_{10}$. Since $r^3 s = sr^{-3} = sr^2 \neq sr^3$, r^3 does not commute with s, so the centraliser is not the whole of $\mathcal{D}_{10}$ and therefore must be $\{1, r, r^2, r^3, r^4\}$.
2. Now instead consider the element r^3 of the dihedral group $\mathcal{D}_{12}$. Find the *centraliser* of r^3 in $\mathcal{D}_{12}$.
   Solution: All rotations commute with r^3, so the centraliser has cardinality at least six. Since in this group $r^3 s = sr^{-3} = sr^3$, r^3 commutes with s and therefore commutes with all elements of $\mathcal{D}_{12}$.
3. Write down the *centre* of the group $\mathcal{D}_{10}$.
   Solution: All rotations (except the identity) fail to commute with s, so the centraliser is just $\{1\}$.

QMplus Quiz

Attempt Quiz 5 at QMplus page

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c, \ ac = ca = b, \ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$