PS4 Q2:

We compute

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) V(x,t)$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) \left[\frac{\partial}{\partial x} \cdot U(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) U(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[U_{tt} - c^2 U_{xx}\right]$$

$$= \frac{\partial}{\partial x} \cdot 0 \quad \leftarrow \text{because } U \text{ satisfies wave equation}$$

$$= 0$$

so V also satisfies wave equation.

PS4 Q5:

The equation can be factored as

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\underbrace{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) U}_{} = 0$$

Denote by $\frac{\partial}{\partial x} + \frac{\partial}{\partial t} U = W$

we get 2 1st order PDEs

$$\begin{cases} \frac{\partial}{\partial x} - 4\frac{\partial}{\partial t} W = 0 & ① \\ \frac{\partial}{\partial x} + \frac{\partial}{\partial t} U = W & ② \end{cases}$$

Equation ① has characteristics $\frac{dt}{dx} = -4$

Namely $t = -4x + c$

So ① becomes an ODE along such characteristics

$$\frac{d}{dx} W(x, t(x)) = W_x + \frac{dt}{dx} W_t$$
$$= W_x - 4W_t$$
$$= 0$$

Thus $W(x,t) = f(c)$

using that $c = 4x+t$, we get

$$W(x,t) = f(4x+t)$$

Next, we solve ②, namely

$$u_x + u_t = f(4x+t) \quad ③$$

This equation has characteristics $\frac{dt}{dx} = 1$

Namely $t = x + \tilde{c}$

Along such characteristics

(and using $t = x + \tilde{c}$), we see

③ becomes an ODE

$$\frac{d}{dx} u(x, t(x)) = f(4x + x + \tilde{c})$$

$$\frac{d}{dx} u = f(5x + \tilde{c})$$

Integrate it with respect to x, we get

$$u = F(5x+\tilde{\tau}) + G(\tilde{\tau})$$

substituting back $\tilde{\tau} = t-x$, we get the general solutions :

$$u(x,t) = F(5x+t-x) + \tilde{G}(t-x)$$

$$= F(4x+t) + G(x-t) \quad (4)$$

for any F, G.

Next we determine F, G by initial conditions

First $u_t(x,t) = F'(4x+t) - G'(x-t)$ (5)

plug in $t=0$ to (4) and (5), we get

$$x^2 = u(x,0) = F(4x) + G(x) \quad (6)$$

$$e^x = u_t(x,t) = F'(4x) - G'(x) \quad (7)$$

Integrate (7) we get

$$e^x = \frac{1}{4}F(4x) - G(x) + D \quad (8).$$

Using (6) and (8), we can solve the algebraic equations and get F, G.

$$\frac{x^2+e^x-D}{\frac{5}{4}} = F(4x)$$

substituting $\frac{x}{4}$ for x, we get

$$F(x) = \frac{4}{5}\left[\frac{x^2}{16}+e^{\frac{x}{4}} - D\right]$$

and $G(x) = -F(4x) + x^2$

$$= \frac{-4}{5}\left(x^2 + e^x - D\right) + x^2$$

$$= +\frac{1}{5}x^2 - \frac{4}{5}e^x + \frac{4}{5}D$$

plug in $x=0$ in ⑥ we get

$$F(0) + G(0) = 0 \quad ,$$

Namely

$$\frac{4}{5}[0+e^0-D] - 0 + \frac{4}{5}e^0 - \frac{4}{5}D = 0$$

so $D = 1$

Thus $F(x) = \frac{4}{5}\left(\frac{x^2}{16}+e^{\frac{x}{4}}-1\right)$

$$G(x) = +\frac{1}{5}x^2 - \frac{4}{5}e^x + \frac{4}{5}$$

plug into the general solutions, get

$$\boxed{u(x,t) = \frac{4}{5}\left[\frac{(4x+t)^2}{16}+e^{\frac{4x+t}{4}}-1\right] + \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t} + \frac{4}{5}}$$

PS4 Q9:

First the general solution for wave equations we deduced in the lecture notes are

$u(x,t) = F(x+ct) + G(x-ct)$

for any F, G

Next we use the boundary conditions to determine F, G.

When $x-ct=0$, Namely $t=\frac{x}{c}$, we have

① $x^2 = u|_{x=ct} = F(x+x) + G(x-x) = F(2x)+G(0)$

when $x+ct=0$, Namely $t=-\frac{x}{c}$, we have

② $x^4 = u|_{x+ct} = F(x-x) + G(x+x) = F(0) + G(2x)$

plugin $x=0$ in ①, we get

③ $F(0) + G(0) = 0$

Replace x by $\frac{x}{2}$ in ①, we get

$$F(x) = \frac{x^2}{4} - G(0)$$

replace x by $\frac{x}{2}$ in 2, we get

$$G(x) = \frac{x^4}{16} - F(0)$$

so the solution is

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{(x+ct)^2}{4} - G(0) + \frac{(x-ct)^4}{16} - F(0)$$

using 3, $F(0) + G(0) = 0$.

so $$u(x,t) = \frac{(x+ct)^2}{4} + \frac{(x-ct)^4}{16} - (F(0)+G(0))$$

$$= \frac{(x+ct)^2}{4} + \frac{(x-ct)^4}{16}$$

Selected solutions to Problem set 4.

1. Using D'Alembert's formula, we get

$$u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

$$= \frac{e^{x+ct} + e^{x-ct}}{2} - \frac{1}{2c}\cos s \Big|_{x-ct}^{x+ct}$$

$$= \cancel{\frac{e^{x+ct} + e^{x-ct}}{2} \quad \frac{1}{2}}$$

$$= e^{x}\,\frac{e^{ct}+e^{-ct}}{2} - \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

$$= e^{x}\cosh ct + \frac{1}{c}\sin x \cdot \sin(ct)$$

2. $V_{tt} = U_{xtt} = U_{ttx}$

$V_{xx} = U_{xxx}$

So $V_{tt} - c^2 V_{xx} = U_{ttx} - c^2 U_{xxx}$

$$= \frac{\partial}{\partial x}\left(U_{tt} - c^2 U_{xx}\right)$$

$$= \frac{\partial}{\partial x} \cdot 0$$

$$= 0$$

So V also satisfies wave equation

4. $\cancel{V_{tt} = \frac{\partial}{\partial t}}$ By Chain Rule.

$$V_t = \partial U_t(\partial x, \partial t)$$

$$V_{tt} = \partial^2 U_{tt}(\partial x, \partial t)$$

$$U_x = \partial U_x(\partial x, \partial t)$$

$$U_{xx} = \partial^2 U_{xx}(\partial x, \partial t)$$

so $V_{tt} - c^2 V_{xx} = \partial^2 (U_{tt} - c^2 U_{xx}) = \partial^2 \cdot 0 = 0$

and V also is a solution to wave equation

7. $$u_t = \frac{c}{2}\left[f'(x+ct) - f'(x-ct)\right] + \frac{1}{2} g(x+ct) + \frac{1}{2} g(x-ct)$$

$$u_{tt} = \frac{c^2}{2}\left[f''(x+ct) + f''(x-ct)\right] + \frac{c}{2} g'(x+ct) - \frac{c}{2} g(x-ct)$$

$$u_x = \frac{1}{2}\left[f'(x+ct) + f'(x-ct)\right] + \frac{1}{2c} g(x+ct) - \frac{1}{2c} g(x-ct)$$

$$u_{xx} = \frac{1}{2}\left[f''(x+ct) + f''(x-ct)\right] + \frac{1}{2c} g'(x+ct) - \frac{1}{2c} g'(x-ct)$$

Combining the above, we get

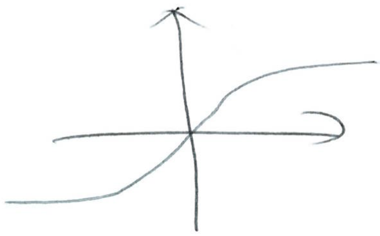
~~Thus~~ $u_{tt} - c^2 u_{xx} = 0$

8. $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{1}{1+s^2} \, ds$

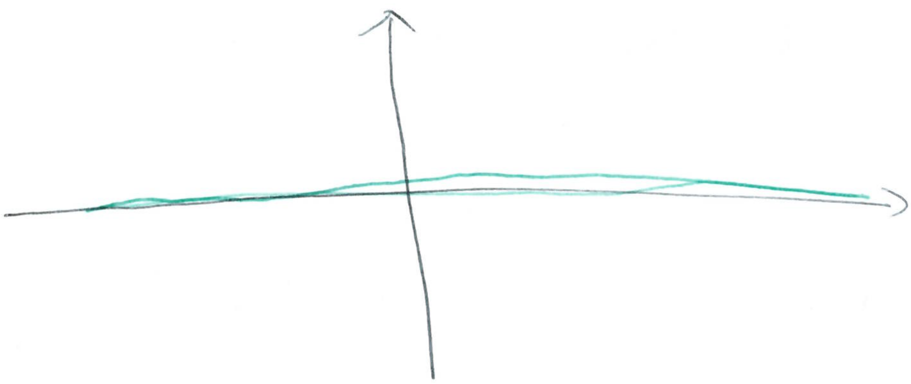
$$= \frac{1}{2c} \arctan x \Big|_{x-ct}^{x+ct}$$

$$= \frac{\arctan(x+ct)}{2c} - \frac{\arctan(x-ct)}{2c}$$

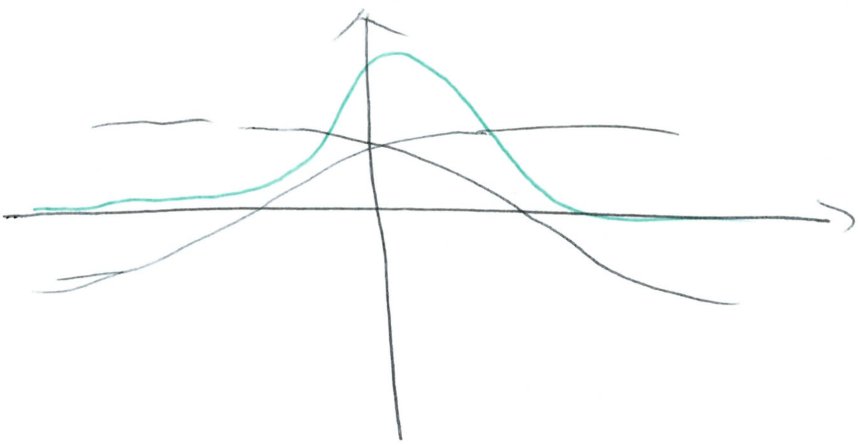
The shape of $\arctan x$ is



At $t=0$



At $t=1$



At $t=5$