Note that for order-preserving diffeomorphisms we cannot make any general statements about the number of fixed points.

Let us look at some examples of order-preserving diffeos:

Example $f(x) = x$

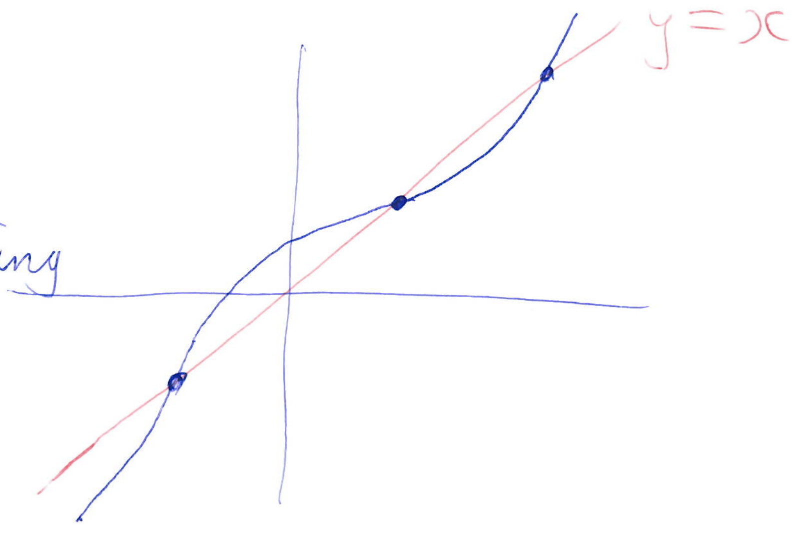
Here every real number is a fixed point.

Example $f(x) = x + c$, where $c \neq 0$.

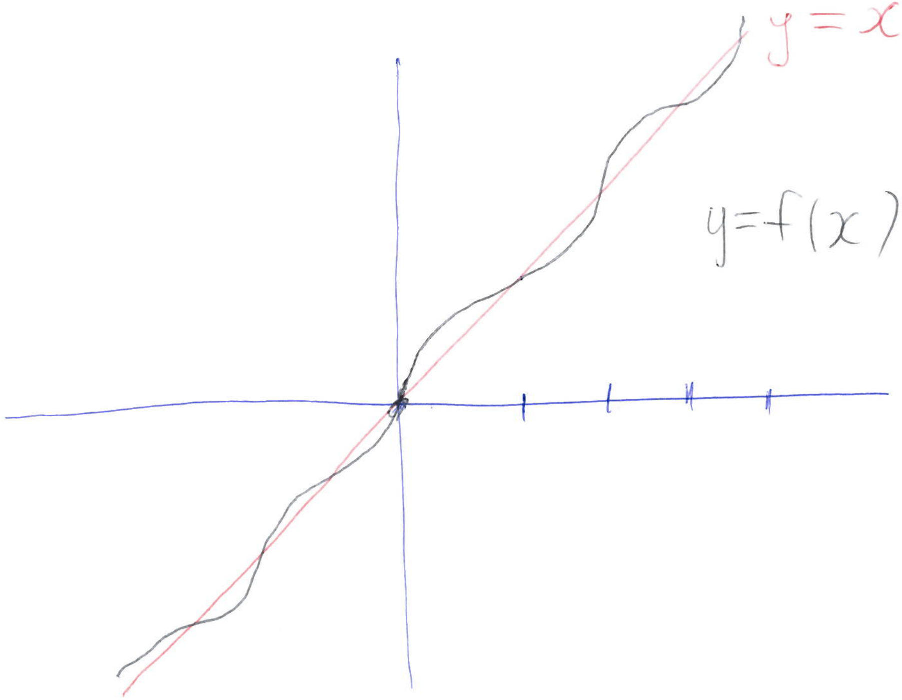
Here there are no fixed points

Example

Illustration of an order-preserving diffeo with precisely 3 fixed points



Example $f(x) = x + \frac{1}{2}\sin(x)$



The fixed point equation is $f(x) = x$

i.e. $x + \frac{1}{2}\sin(x) = x$

i.e. $\frac{1}{2}\sin(x) = 0$

So $x = n\pi$, $n \in \mathbb{Z}$, are fixed points.

Note that $f'(x) = 1 + \frac{1}{2}\cos x$

$$\geq 1 - \frac{1}{2}$$

$$= \frac{1}{2} > 0 \quad \text{for all } x \in \mathbb{R}$$

So f is an order-preserving diffeomorphism

We can check if the fixed points at $x = n\pi$, $n \in \mathbb{Z}$, are attracting or repelling.

$$f'(n\pi) = 1 + \tfrac{1}{2}\cos(n\pi)$$

$$= \begin{cases} 3/2 & \text{if } n \text{ is even} \\ 1/2 & \text{if } n \text{ is odd} \end{cases}$$

So (by the Theorem proved last week), the fixed point $n\pi$ is repelling if n is even (since $|f'(n\pi)| > 1$) and attracting if n is odd (since $|f'(n\pi)| < 1$).

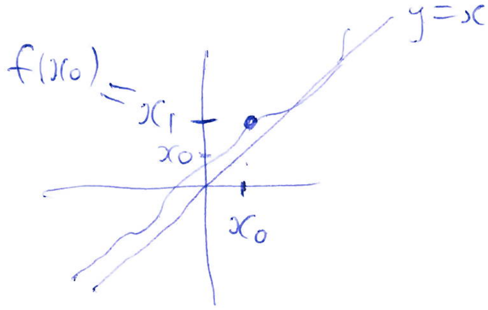
Periodic points of higher period

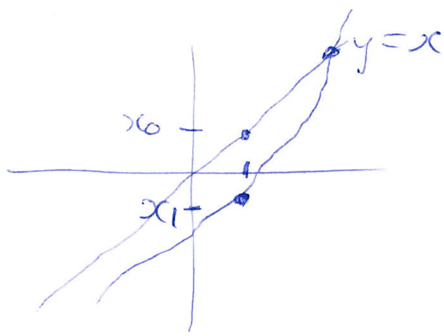
Proposition If $f : \mathbb{R} \to \mathbb{R}$ is an order-preserving diffeomorphism then any periodic point for f must be a fixed point.

(i.e. There are no points of least period k for $k > 1$)

Proof Let $x_0 \in \mathbb{R}$. As usual, let $x_n = f^n(x_0)$.

If x_0 is not a fixed point then either $x_1 > x_0$ or $x_1 < x_0$





If $x_0 < x_1$ then $f(x_0) < f(x_1)$

i.e. $x_1 < x_2$

But then apply f again to get:

$f(x_1) < f(x_2)$

i.e. $x_2 < x_3$

So $x_0 < x_1 < x_2 < x_3 < \ldots$

So in particular, $x_k \neq x_0$ for all $k \geq 1$,

therefore x_0 is not periodic

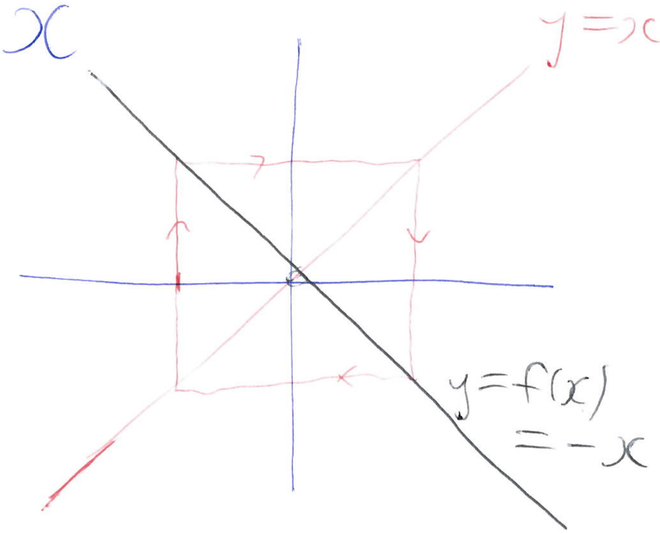
Similarly, if $x_0 > x_1$ then the order-preserving property, applied repeatedly, implies $x_0 > x_1 > x_2 > x_3 > \ldots$

and again we see that x_0 is not periodic. $\square$

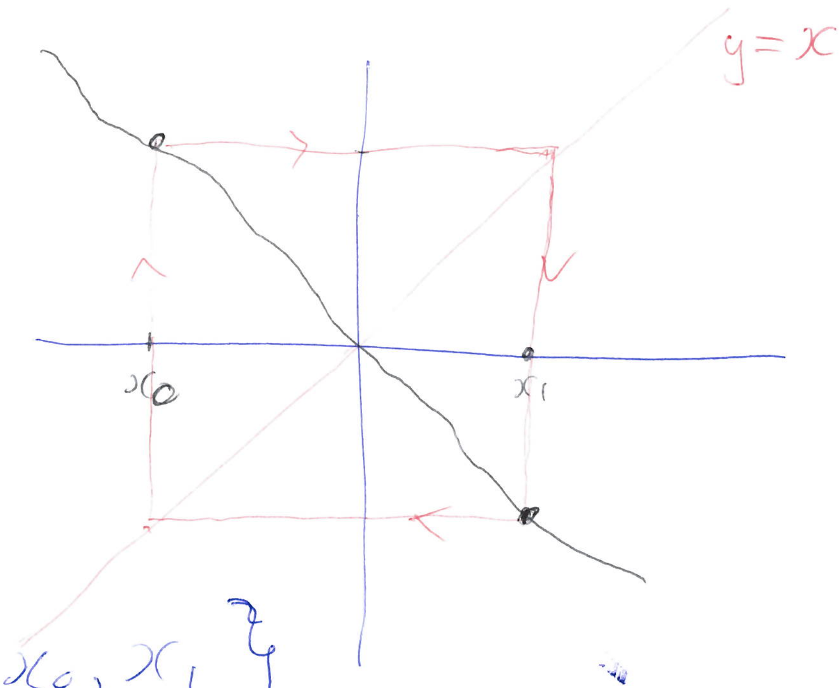
What about periodic points for order-<u>reversing</u> diffeomorphisms?

Example $f(x) = -x$

Here, 0 is the unique fixed point, but every other point has least period 2

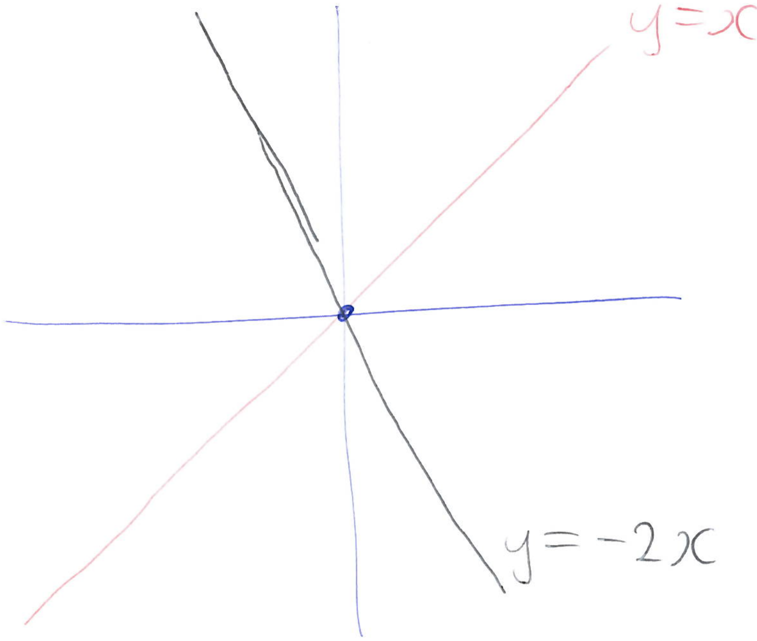


Example
Illustration of another order-reversing diffeomorphism with a 2-cycle $\{x_0, x_1\}$



Example

$$f(x) = -2x$$



Here there are no points of least period 2.

As we see with these examples, for order-reversing diffeomorphisms there could be points of least period 2, but there need not be.

Rather like the situation for fixed points of order-preserving diffeomorphisms, we will not prove any general result about least period 2 points for order-reversing diffeomorphisms.

However, there is an important result about points of least period > 2:

<u>Proposition</u> If $f : \mathbb{R} \to \mathbb{R}$ is an order-reversing diffeomorphism then there are <u>no</u> points of least period strictly larger than 2.

<u>Proof</u> Since f is a diffeomorphism so is $f^2 = f \circ f$. Notice that f^2 is is order-<u>preserving</u> because

$$(f^2)'(x) = \underbrace{f'(f(x))}_{<0} \cdot \underbrace{f'(x)}_{<0}$$

which is > 0 (since $f' < 0$, and the product of two negative values is positive)

So by the previous Proposition, f^2 does not have any periodic points of least period > 1.

So f itself does not have any periodic points of least period $2m$ for $m > 1$, i.e. f has no points of least period k for <u>even</u> numbers $k > 2$.

To address the case where k is odd (and ≥ 3) we note that for such k, the map f^k is an order-reversing diffeomorphism.

But order-reversing diffeomorphisms have <u>precisely</u> one fixed point, and of course the unique fixed point p for f is also a fixed point for f^k

therefore p is the unique fixed point for f^k.

So f^k has no other fixed points.

In other words, there are no period-k points for f (except for the fixed point p).

i.e. There are no points of least period k. $\square$

Summarising the situation for all diffeomorphisms:

	Fixed Points	Least Period 2	Least Period > 2
f order-preserving	Arbitrary number	None	None
f order-reversing	Exactly one	Arbitrary number	None

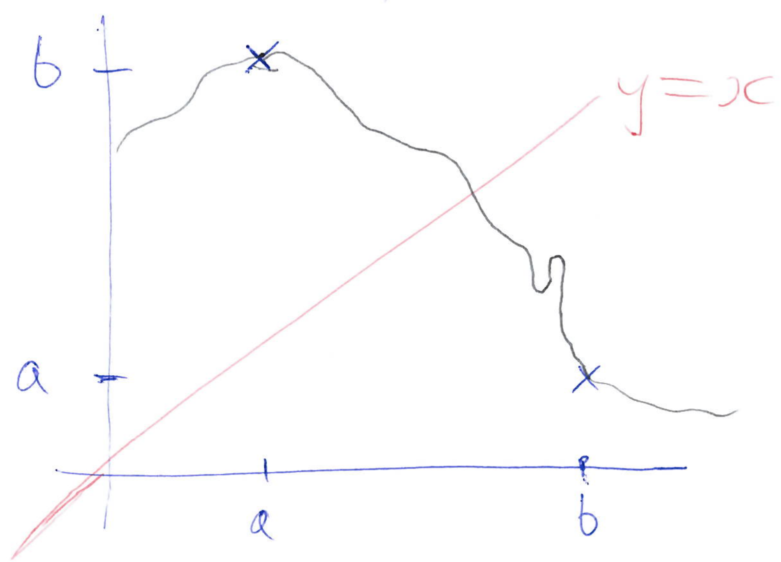
Dynamics of continuous maps f

— fixed points + periodic points

Proposition Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. If f has an orbit of least period 2 then it has a fixed point.

Proof Let $\{a, b\}$ be a 2-cycle for f, with $a < b$.

Then $f(a) = b$, and $f(b) = a$.



Let $g(x) := f(x) - x$, so that a zero of g (i.e. a 'root' of g) is a fixed point of f.

Clearly f is continuous on $[a,b]$ since it is continuous on $\mathbb{R}$, therefore ~~Also~~ g is also continuous on $[a,b]$.

Also,
$$g(a) = f(a) - a$$
$$= b - a > 0,$$
and
$$g(b) = f(b) - b$$
$$= a - b < 0$$

By the Intermediate Value Theorem (applied to the continuous function g) there exists $c \in (a,b)$ such that $g(c) = 0$, i.e. such that $f(c) = c$.

So f has a fixed point c. $\square$

<u>Remark</u> This is an example of "forcing", i.e. the presence of a periodic orbit of some period <u>forces</u> the presence of an orbit of another period.

[Here: "Period 2 $\Rightarrow$ Period 1"]

Perhaps the most famous result of this type is the "Period-3 implies chaos" theorem, which more precisely states:

<u>Theorem</u> If $f: \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit of least period 3, then it has periodic orbits of <u>all</u> other least periods n, for $n \in \mathbb{N}$.

This "period-3 implies chaos" theorem is actually a consequence of a more general theorem due to Sharkovskii.

<u>Definition</u> The <u>Sharkovskii order</u> of the natural numbers $\mathbb{N}$ is given by:

$$1 \lhd 2 \lhd 4 \lhd 8 = 2^3 \lhd 2^4 \lhd \ldots \lhd 2^n \lhd \ldots$$

$$\vdots$$

$$\ldots \lhd 2^k(2n+1) \lhd \ldots \lhd 2^k \times 7 \lhd 2^k \times 5 \lhd 2^k \times 3$$

$$\vdots$$

$$\ldots \lhd 2^2(2n+1) \lhd \ldots \lhd 2^2 \times 7 \lhd 2^2 \times 5 \lhd 2^2 \times 3$$

$$\ldots \lhd 2(2n+1) \lhd \ldots \lhd 2 \times 7 \lhd 2 \times 5 \lhd 2 \times 3$$

$$\ldots \lhd (2n+1) \lhd \ldots \lhd 9 \lhd 7 \lhd 5 \lhd 3$$

Sharkovskii's Theorem

Let $I = (a,b) \subseteq \mathbb{R}$ (we allow $I = \mathbb{R}$
$= (-\infty, \infty)$)

Suppose $f: I \to I$ is continuous, and has an m-cycle (i.e. a periodic orbit of least period m).

Then f has an n-cycle for all those natural numbers n for which

$$n \lhd m.$$

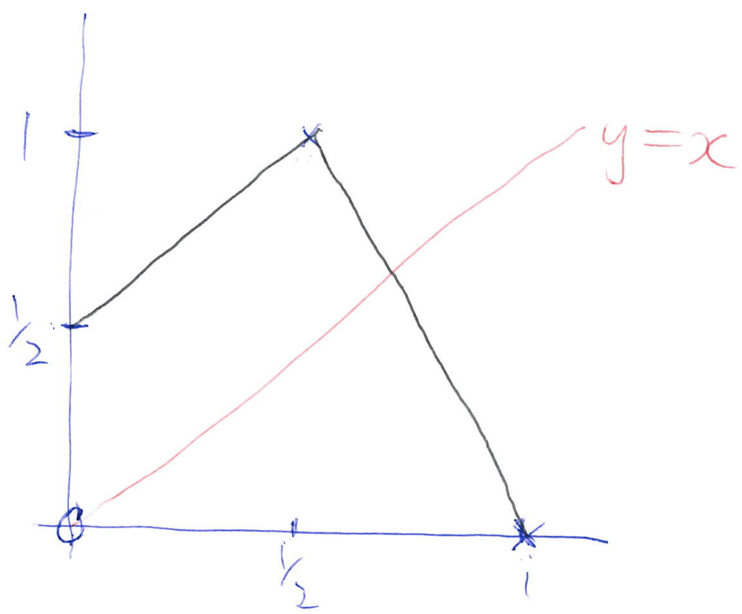
(where $\lhd$ is the Sharkovskii order on $\mathbb{N}$)

__Example__ Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit of least period $8 = 2^3$. Then by Sharkovskii's Theorem, f has a fixed point, and a 2-cycle, and a $\underbrace{4}_{=2^2}$-cycle.

Example Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and has a 10-cycle. Then by Sharkovskii's Theorem, f has n-cycles for all even numbers n except $n = 6$.

Example Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$



Fixed points are given by $f(x) = x$

Clearly $x + 1/2 = x$ has <u>no solutions</u>, so it suffices to consider $2 - 2x = x$,

i.e. $2 = 3x$, i.e. $x = 2/3$

Period-2 orbits are given by solutions to the equation $f^2(x) = x$.

Assume $x \in [0, 1/2]$. Solve $f^2(x) = x$

i.e. $f(f(x)) = x$

i.e. $f(x + 1/2) = x$ — this step is justified because $x + 1/2 \in [1/2, 1]$, so $f(x+1/2) = 2 - 2(x + 1/2)$

i.e. $2 - 2(x + 1/2) = x$

i.e. $2 - 2x - 1 = x$

i.e. $1 = 3x$

So $x = 1/3$ has period 2 (indeed it has <u>least</u> period 2)

Therefore also $f(1/3) = 1/3 + 1/2 = 5/6$ is <u>also</u> a point of least period 2.

So $\{1/3, 5/6\}$ is a 2-cycle

~~This~~ There is also a 3-cycle,

namely $\{0, 1/2, 1\}$

(Since
$$\begin{tikzcd} 0 \arrow[r, mapsto, "f"] & 1/2 \arrow[r, mapsto, "f"] & 1 \arrow[ll, bend left, "f"] \end{tikzcd}$$
)

Therefore, by Sharkovskii's Theorem this f has n-cycles for all $n \in \mathbb{N}$.