Group Theory

Week 4, Lecture 1, 2 & 3

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Definition of Cosets

Definition

Let H be a subgroup of the group G whose operation is written multiplicatively (juxtaposition denotes the group operation). Given an element g of G, the left cosets of H in G are the sets obtained by multiplying each element of H by a fixed element g of G (where g is the left factor).

$gH = \{gh : \text{ h an element of H}\}$ for g in G.

The right cosets are defined similarly, that is,

$Hg = \{hg : \text{ h an element of H}\}$ for g in G.

Cosets: Example

Example: Let G be the dihedral group of order six. Its elements may be represented by $\{I, a, a^2, b, ab, a^2b\}$. In this group, $a^3 = b^2 = I$ and $ba = a^2b$. This is enough information to fill in the entire Cayley table:

*	**I**	**a**	**a²**	**b**	**ab**	**a²b**
I	I	a	a^2	b	ab	a^2b
a	a	a^2	I	ab	a^2b	b
a²	a^2	I	a	a^2b	b	ab
b	b	a^2b	ab	I	a^2	a
ab	ab	b	a^2b	a	I	a^2
a²b	a^2b	ab	b	a^2	a	I

Let **T** be the subgroup $\{I, b\}$. The left cosets of T are: $IT = T = \{I, b\}$, $aT = \{a, ab\}$, and $a^2T = \{a^2, a^2b\}$. The right cosets of T are: $TI = T = \{I, b\}$, $Ta = \{a, ba\} = \{a, a^2b\}$ and $Ta^2 = \{a^2, ba^2\} = \{a^2, ab\}$.

Let **H** be the subgroup $\{I, a, a^2\}$. The left cosets of H are $IH = H$ and $bH = \{b, ba, ba^2\}$. The right cosets of H are $HI = H$ and $Hb = \{b, ab, a^2b\} = \{b, ba^2, ba\}$. In this case, every left coset of H is also a right coset of H? Why?

Cosets: Example

Example: Take $G = \mathcal{S}_3$, $H = <(12)> = \{id, (12)\}$ and $g = (23)$. Then $Hg = \{(23), (123)\}, gH = \{(23), (132)\}$.

Remark:
(i) H is always a right coset of itself, since

$$H1 = \{h1 | h \in H\} = \{h | h \in H\} = H.$$

(ii) We can have $Hf = Hg$ even when $f \neq g$.

For example, let $G = \mathcal{C}_6 = \{1, z, z^2, z^3, z^4, z^5\}$. Then $H = \{1, z^3\}$ is a subgroup. We have $Hz = \{z, z^4\}$, and also $Hz^4 = \{z, z^4\}$. We can see that $Hf = Hg$.

Cosets: Proposition

Proposition

Suppose G is a group, $H \leq G$ and $f, g \in G$.

1. $|Hg| = |H|$.
2. If $f \in Hg$, then $Hf = Hg$.
3. Each element of G is contained in exactly one right coset of H.

Cosets: Proposition

Cosets: Proposition

Example

Take $G = D_8$ and $H = \{1, rs\}$. Then the right cosets are

$$H1 = \{1, rs\}$$
$$Hr = \{r, s\}$$
$$Hr^2 = \{r^2, r^3 s\}$$
$$Hr^3 = \{r^3, r^2 s\}$$

Cosets: Coset Lemma

Coset lemma

Suppose G is a group, $H \leq G$ and $f, g \in G$. Then:
(i) Hf = Hg if and only if $fg^{-1} \in H$;
(ii) fH = gH if and only if $f^{-1}g \in H$.

Cosets

Proposition

If G is a group and $H \leq G$, then the number of right cosets of H is equal to the number of left cosets of H.

Cosets

Lagrange's Theorem

Definition

Suppose G is a group and $H \leq G$. The index of H in G is the number of right cosets of H in G, written as $|G : H|$.

Lagrange's Theorem

Suppose G is a group and $H \leq G$. Then $|G| = |H||G : H|$. In particular, if G is finite then $|H|$ divides $|G|$.

Proof of Lagrange's Theorem

Lagrange's Theorem

Example: $G = \mathbb{Z}$, $H = 3\mathbb{Z}$.

$$[G : H] = [\mathbb{Z} : 3\mathbb{Z}] = 3$$

The cosets of H in $\mathbb{Z}$ are of the form:

$$g + H = \{g + h | h \in H\} = \{g + 3k | k \in \mathbb{Z}\}$$

where $g \in \mathbb{Z}$.
Thus the distinct cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ are:
(i) $0 + 3\mathbb{Z} = \{0, 3, 6, 9, \cdots\}$
(ii) $1 + 3\mathbb{Z} = \{1, 4, 7, 10, \cdots\}$
(iii) $2 + 3\mathbb{Z} = \{2, 5, 8, 11, \cdots\}$

Lagrange's Theorem

Example Dihedral group $\mathcal{D}_8$:

Consider the dihedral group $\mathcal{D}_8$, which represents the symmetries of a square. The order of 8. The elements of $\mathcal{D}_8$ consist of 4 rotations and 4 reflections.
Let $H = \{e, r, r^2, r^3, \}$, where e is the identity and r, r^2, r^3 are the rotations by 90°, 180°, and 270°. This is a subgroup of $\mathcal{D}_8$ (the group of rotations), and its order is 4 Since 4 divides 8, Lagrange's Theorem holds for this example.
No of cosets of H in $\mathcal{D}_8$ are 2, which are

$$H = \{1, r, r^2, r^3\} \text{ and } Hs = \{s, rs, r^2s, r^3s\}$$

Symmetric Group $\mathcal{S}_3$: Consider the symmetric group $\mathcal{S}_3$, which is the group of all permutations of 3 elements. The order of $\mathcal{S}_3$ is 6. The elements of $\mathcal{S}_3$ are:
$\mathcal{S}_3 = \big\{e, (12), (13), (23), (123), (132)\big\}$. Now, consider the subgroup $H = \big\{e, (12)\big\}$. This is a subgroup of $\mathcal{S}_3$, and the order of H is 2. According to Lagrange's Theorem, the order of H must divide the order of S_3, which is true. The list of right cosets of H is

$$H = \{e, (12)\}, \{(13), (132)\}, \text{ and } \{(23), (123)\}$$

Corollary

Corollary

Suppose G is a finite group and $g \in G$. Then ord(g) divides $|G|$.

Euler's Theorem

Applications in Number Theory

$$\mathcal{U}_n = (\mathbb{Z}/n\mathbb{Z})^\times$$

Invertible residues mod n.
$|\mathcal{U}_n| = \Phi(n) =$ no of residues coprime to n.

Pick a residues $x \in \mathcal{U}_n$ it generates a cyclic subgroup

$$\langle x \rangle = \left\{1, x, x^2, \cdots x^{ord(x)-1}\right\}$$

of order ord(g).
Ord(g) $|\Phi(n) \implies x^{\Phi(n)} = 1$.

In other words
If x is coprime to n, then $x^{\Phi(n)} \equiv 1 \, mod(n)$.

Euler's Theorem

Example:

$$3^4 = 81 \equiv 1 \text{ mod } (8)$$

$$\varphi(8) = |\{1, 3, 5, 7, \}| = 4$$

If $n = p$ is prime, then $\varphi(n) = p - 1$, $X^{p-1} \equiv 1 \text{mod } p \Leftrightarrow X^P - X$ is divisible by p.

For example $X^7 - X$ is divisible by 7, $2^7 - 2 = 128 - 2 = 126 = 7.18$.

Lagrange's Theorem

Corollary: If $g \in G$ then $\text{or}(g) || G|$.

Example

S_n is symmetric group. $|S_n| = n!$. , $\mathcal{A}_n =< \mathcal{S}_n$ group of even permutation.

$$[\mathcal{S}_n : \mathcal{A}_n] = 2$$

Which right cosets do we know.

$$\mathcal{A}_n \text{ all the even permutation}$$

$$\text{Coset of } \mathcal{A}_n(12) = \text{ odd permutations}$$

$$[\mathcal{S}_n : \mathcal{A}_n] = \frac{|\mathcal{S}_n|}{|\mathcal{A}_n|}$$

Conjugacy

Definition

Suppose G is a group and $f, g \in G$. We say that f is conjugate to g in G (written $f \sim_G g$) if there is $k \in G$ such that $kfk^{-1} = g$.

Lemma

Suppose G is a group then $\sim_G$ is an equivalence relation.

Proof.

1. Reflexive: $x = exe^{-1}$
2. Symmetric: $x = gyg^{-1} \implies y = g^{-1}xg$.
3. Transitive: $x = gyg^{-1}$ and $y = hzh^{-1} \implies x = (gh)z(gh)^{-1}$

□

Conjugacy

Conjugacy in $\mathcal{D}_8$

Example: **Dehidral group of order** 8 In this case the conjugacy classes

correspond to "types of symmetry":

- r and r^3 are both 90° rotations;
- r^2 is the only 180° rotation;
- s and r^2s are both reflections in axes parallel to the sides of the square;
- rs and r^3s are both reflections in diagonals of the square.

Conjugacy classes in D_8

Conjugacy classes in $\mathcal{D}_8$ are

$$\left\{\{1\}, \{r, r^3\}, \{r^2\}, \{s, r^2s\}, \{rs, r^3s\}\right\}$$

The elements r and r^3 are conjugate to each other. Reflections s and sr conjugate r to r^3, but r and r^3 are not conjugate to other elements of the group.

Conjugacy

$$srs^{-1} = r^3$$

$$rsr^{-1} = sr^2$$
$$rsrr^{-1} = sr^3$$

The element r^2 is in its own conjugacy class.

Conjugacy

Conjugacy classes in D_{10}

Exercise: Find the conjugacy classes of D_{10}.

Elements of D_{10} can be written as:
Rotations: $\{e, r, r^2, r^3, r^4\}$ where r represents a rotation by $2\pi/5(72^\circ)$ degree, and e is the identity (rotation by 0 degrees).

Reflections: $\{s, sr, sr^2, sr^3, sr^4\}$ where each s represents a reflection across a line of symmetry through a vertex or an edge.

W can write $D_{10} = \left\{ \{e\}, \{r^2\}, \{r, r^3\}, \{s, r^2s\}, \{r, r^3s\} \right\}$

Conjugacy

It turns out that:
Conjugacy class of $r = \{r, r^4\}$, $srr^{-1} = r^4$
Conjugacy class of $r^2 = \{r^2, r^3\}$, $sr^2s^{-1} = r^4$
The rotations split into two conjugacy classes: $\{r, r^2\}$ and $\{r^2, r^3\}$ as the as the elements r and r^4 are conjugate to each other, and similarly r^2 and r^3 are conjugate to each other.

Reflections: All reflections in D_{10} are conjugate to each other. Conjugating any reflection by a rotation produces another reflection:
Conjugacy class of reflections $= \{s, sr, sr^2, sr^3, sr^4\}$
This means all 5 reflections form a single conjugacy class.
Thus, the group D_{10} has four distinct conjugacy classes.

1) $\{e\}$ (the identity element), 2) $\{r, r^4\}$ (rotations by 72° and 288°), 3) $\{r^2, r^3\}$ (rotations by 144° and 216°
4) $\{s, sr, sr^2, sr^3, sr^4\}$ (all reflections)

Centre of a Group

Definition

If G is a group, the **centre** of G is

$$Z(G) = \left\{ g \in G | hg = gh \text{ for all } h \in G \right\}.$$

- Suppose G is abelian. Then $hg = gh$ for all g, h, so $Z(G) = G$.
- Let's find the centre of
  $\mathcal{Q}_8 = \left\{ 1, -1, i, -i, j, -j \right\}$.
  Certainly $1 \in Z(\mathcal{Q}_8)$, because $1g = g = g1$ for all g. Also $-1 \in Z(\mathcal{Q}_8)$, because -1 changes the sign of everything whether we multiply it from the left or the right.
  $ij \neq ji$, which means that neither i nor j is in $Z(\mathcal{Q}_8)$. Similarly we can show $-i, -j, k, -k \notin Z(\mathcal{Q}_8)$. So $Z(\mathcal{Q}_8) = \{1, -1\}$.

Centre of a Group

Proposition

If G is a group, then $Z(G) \leq G$.

Solution:

- For any $h \in G$ we have
$$h1 = h = 1h,$$
so $1 \in Z(G)$.
- Suppose $f, g \in Z(G)$. Then for any $h \in G$
$$hfg^{-1} = fhg^{-1} = fg^{-1}ghg^{-1} = fg^{-1}hgg^{-1} = fg^{-1}h,$$
so $fg^{-1} \in Z(G)$.

Centre of a Group

Lemma

Suppose G is a group and $x \in G$. Then $x \in Z(G)$ if and only if x lies in a conjugacy class of itself.

Proof.

Suppose x is in its own conjugacy class. This means that

$$gxg^{-1} = x, \quad \forall g \in G \Leftrightarrow gx = xg, \quad \forall g \in G \leftrightarrow x \in Z(G)$$

□

Conjugacy in $\mathcal{S}_n$

Proposition

Suppose $n \geq 3$. Then $Z(\mathcal{S}_n) = \{id\}$.

Proof:

We know id $\in Z(\mathcal{S}_n)$, so we just need to show that if $g \in \mathcal{S}_n$ and $g \neq$ id then $g \notin Z(\mathcal{S}_n)$, i.e. there is some $h \in \mathcal{S}_n$ such that $gh \neq hg$.

Since $g \neq$ id, we can find $a \neq b \in \{1, \ldots, n\}$ such that $g \cdot a = b$. Let $c \in \{1, \ldots, n\}$ be different from a and b, and let $h = (b\,c)$. Then

$$gh \cdot a = g \cdot a = b, \qquad hg \cdot a = h \cdot b = c,$$

so $gh \neq hg$.

Conjugacy in $\mathcal{S}_n$

Definition

Suppose $f \in \mathcal{S}_n$, written in disjoint cycle notation. The **cycle type** of f is the list of the lengths of the cycles of f, written in decreasing order.

Example:

In $\mathcal{S}_9$, the permutation $(1\,4\,3)(2896)$ has cycle type $(4, 3, 1, 1)$. Notice in particular that the cycle lengths must be written in decreasing order, and we include cycles of length 1 (even though we usually don't write them when we're writing down the permutation).

Theorem

Suppose $f, g \in \mathcal{S}_n$. Then $f \sim_{\mathcal{S}_n} g$ if and only if f and g have the same cycle type.

Big Idea

Conjugate permutations have the same structure. Such permutations are the same up to renumbering.

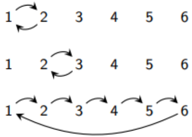
Conjugacy in $\mathcal{S}_n$

Consider the following permutations in $G = S_6$:

$$g = (1\ 2)$$

$$h = (2\ 3)$$

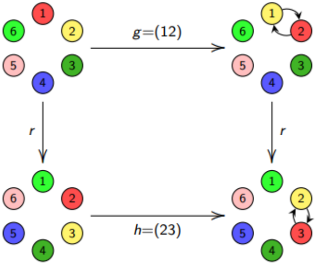
$$r = (1\ 2\ 3\ 4\ 5\ 6)$$



Since g and h have the same cycle type, they are conjugate:

$$(1\ 2\ 3\ 4\ 5\ 6)\,(2\ 3)\,(1\ 6\ 5\ 4\ 3\ 2) = (1\ 2)\,.$$

Here is a visual interpretation of $g = rhr^{-1}$:



Conjugacy in $\mathcal{S}_n$

Theorem

Suppose $f, g \in \mathcal{S}_n$. Then $f \sim_{\mathcal{S}_n} g$ if and only if f and g have the same cycle type.

Proof:

Conjugacy in $\mathcal{S}_n$

Exams Style Questions

Conjugacy classes of $\mathcal{S}_3$:

$$\left\{ id, \{(12),(23),(13)\}, \{(123),(213)\} \right\}$$

xmxma

Conjugacy classes of $\mathcal{S}_4$:

$$\left\{ id, \{(12),(23),(13),(14)\}, \{(123),(213),413\} \cdots \right\}$$

id, transpositions, (12)(34) , 3 -cycles, 4 -cylces

Exams Style Questions

Example: In $\mathcal{S}_3$ the elements (1 2 3) and (1 3 2) are conjugate.

Transpositions are conjugate: $\{(12), (13), (23)\}$.

Identity element is a conjugate class.

Exams Style Questions

Example: In $\mathcal{S}_9$, find g such that $g = kfk^{-1}$

$f = (1356)(28)(497)$ and

$$k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \downarrow & \downarrow & \downarrow & \downarrow & & & & \\ 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 9 \end{pmatrix}$$

$$g = kfk^{-1} = (4682)(15)(397)$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \downarrow & \downarrow & & & & & & \\ 5 & 4 & 9 & 6 & 1 & 8 & 3 & 2 & 7 \end{pmatrix}$$

Exams Style Questions

What are the conjugacy classes of $\mathcal{Q}_8 = \{1, -1, i, -i, j, -j, k, -k\}$.

Solution:

1. $\{1\}$
2. $\{-1\}$
3. $\{i, -i\}$
4. $\{j, -j\}$
5. $\{k, -k\}$

Exams Style Questions

Question: Write few elements of Conjugacy class of (13)(4679) in $\mathcal{S}_9$

Solution: Here is the list of some elements that belong to conjugacy class of (13)(4679).

1. (2 4)(5781) transposition (24) and 4 -cycle (5781)
2. (56)(2983)
3. (18)(2347)

Exams Style Questions

Question: In this question we work with the group
$\mathcal{U}_{21} = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$.

1. Find order of 5.
2. Hence find an element of order 3, and a subgroup H of order 3.
3. Find all the right cosets of H in $\mathcal{U}_{21}$.

Exams Style Questions

Solution:

1. We calculate $5^2 = 4$, $5^3 = 20$, $5^4 = 16$, $5^5 = 17$, $5^6 = 1$, so order of $5 = 6$.
2. Since 5 has order 6, $5^2 = 4$ has order 3. So $H = \langle 4 \rangle = \{1, 4, 16\}$ is a subgroup of order 3.
3. $$H1 = \{1, 4, 16\},$$ $$H5 = \{5, 20, 17\},$$ $$H2 = \{2, 8, 11\},$$ $$H10 = \{10, 19, 13\}.$$

   (We know we've found all the right cosets because we've written each element of $\mathcal{U}_{21}$ once.)

Exams Style Questions

Question:

Consider the following permutations:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 8 & 4 & 3 & 6 & 7 & 5 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 1 & 3 & 2 & 5 & 6 & 8 & 4 \end{pmatrix}.$$

Write down the disjoint cycle notation for a and b, and also for ab, ba, a^{-1}, b^{-1}, $(aba)^{-1}$ and $b^{-1}ab$. Which of these permutations lie in $\mathcal{A}_8$?

$$\begin{aligned} a &= (128)(34)(567) \\ b &= (17842) \\ ab &= (1567)(348) \\ ba &= (243)(5687) \\ a^{-1} &= (182)(34)(576) \\ aba^{-1} &= (13825) \\ b^{-1} &= (12487) \\ b^{-1}ab &= (156)(247)(38). \end{aligned}$$

QMplus Quiz

Attempt Quiz 4 at QMplus page

Some Useful Notations

Throughout this course, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- Klein group often symbolized by the letter $\mathcal{V}_4$ or as $K_4 = \mathbb{Z}_4 \times \mathbb{Z}_4$ denotes the group $\{1, a, b, c\}$, with group operation given by

$$a^2 = b^2 = c^2 = 1, \qquad ab = ba = c,\ ac = ca = b,\ bc = cb = a.$$

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.

Some Useful Notations

- $\mathcal{D}_{2n}$ is the group with $2n$ elements

  $$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $GL_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

  $$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$